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KIRIRI WOMENS' UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATION, 2023/2024 ACADEMIC YEAR
FOURTH YEAR, SECOND SEMESTER EXAMINATION
FOR THE DEGREE OF BACHELOR OF SCIENCE
(MATHEMATICS)

Date:
Time: 11.30am –1.30pm

KMA 413 - INTRODUCTION TO STOCHASTIC PROCESS

INSTRUCTIONS TO CANDIDATES

ANSWER **QUESTION ONE (COMPULSORY)** AND **ANY OTHER TWO** QUESTIONS

QUESTION ONE (30 MARKS)

a) Five hundred people travelled from City A to City B. Half travelled by airplane and half by bus. On the return trip, 10% of those who travelled by airplane switched to a bus and of those who travelled by bus, 20% switched to a plane. How many travelled by bus on the return trip?
(4 marks)

b) In a Bank, customers arrive to deposit cash to a single counter server every 15 minutes. The bank staff on an average takes 10 minutes to serve a customer. The manager of the bank noticed that on an average at least one customer was waiting at the counter. To eliminate the customer waiting time, the manager provided an automatic currency counting machine to the staff. This decreased the service time to 5 minutes on an average to every customer. Determine whether this rate of service will satisfy the manager's interest.
(5 marks)

c) Research students studying population characteristics found out by their survey that annual mobility of the population, in percent, of a state to a village (V), town (T), and city (C) is in the following percentages

$$\begin{array}{c} \\ V \\ T \\ C \end{array} \begin{array}{c} \begin{array}{ccc} V & T & C \end{array} \\ \begin{bmatrix} 50 & 30 & 20 \\ 10 & 70 & 20 \\ 10 & 40 & 50 \end{bmatrix} \end{array}$$

If it was found that after two (2) years, the total population is 20 million, determine the population sizes in village, town and city after the said duration given that the present population has proportions of 0.7, 0.2 and 0.1 respectively (5 marks)

d) Define the term Markov chain as used in stochastic process and give any two properties of Markov chains (3 marks)

e) Determine the generating function of the sequence of squares $\{0^2, 1^2, 2^2, 3^2, ...k^2\}$
(5 marks)

f) Find the stationary distribution for the transitional matrix below

$$\begin{bmatrix} 0 & 3/4 & 1/4 \\ 0 & 0 & 1 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

(4 marks)

g) Let $X_i = 1$ if it rains in day i otherwise $X_i = 2$. Suppose $P_{11} = 0.8$ and $P_{12} = 0.3$, find the probability that it will rain on Thursday if rains on Tuesday (4 marks)

QUESTION TWO (20 MARKS)

a) Let $\{X_n, \ n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and with transition matrix

$$\begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$ and the initial distribution $\Pr\{X_0 = i\} = 1/3, \quad i = 0,1,2$

i) Find $\Pr\{X_2 = 2, X_1 = 1 / X_0 = 2\}$ (3 marks)

ii) Hence or otherwise find $\Pr\{X_2 = 2, X_1 = 1, X_0 = 2\}$ (2 marks)

b) Consider the generating function $\dfrac{1}{(s+3)(s-7)}$.

i) Express the generating function in partial fraction (2 marks)

ii) Find the coefficient of the kth term in the sequence obtained from the generating function (4 marks)

iii) Use the expression of the coefficient in (b) ii) above to find a_k for $k = 0, 1, 2$ (3 marks)

c) Consider a modified geometric distribution given by $\Pr(X = x) = q^{k-1}p, \quad k = 1, 2, \ldots\ldots$

i) Show that the probability generating function of X is given by $G(S) = \dfrac{ps}{1-qs}$. (2 marks)

ii) Further, show that $E(X) = 1/p$ and $Var(X) = q/p^2$ (4 marks)

QUESTION THREE (20 MARKS)

a) Let $Z(t)$ represent population size at time t and $P_n(t)$ be the probability that a population is of size n at time t. Further let Δt represent a small time interval over which the population is studied. Assuming that $\lambda_n \Delta t + 0\Delta t$ is the probability that a birth occurs within the time interval and Δt and $N_n \Delta t + 0\Delta t$ be the probability that from the population of size n, a death occurs. Given that the difference differential equations (dde) is

$$P_n'(t) = -(\lambda_n + N_n)P_n(t) + \lambda_{n-1} P_{n-1}(t) + N_{n+1} P_{n+1}(t), \qquad n \geq 1 \text{ and}$$

$$P_n'(t) = N_0 P_0(t) + N_1 P_1(t), \qquad n = 0$$

Consider a Poisson process when $N_n = 0$, obtain the corresponding difference differential equations. Further, using the initial conditions, $P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$,

i) Derive the probability generating function of the process

(7 marks)

ii) Hence or otherwise, find the mean and variance of the process

(4 marks)

b) Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean of 3 minutes

i) What is the probability that a person arriving at the booth will have to wait?

(3 marks)

ii) The telephone department will install a second booth when convinced that an arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify the second booth?

(3 marks)

iii) What is the probability that an arrival will take more than 10 minutes all together to wait for the phone and complete his call? (3 marks)

QUESTION FOUR (20 MARKS)

a) Consider a cyclical random walk process with n states such that

$\Pr(E_i \to E_{i+1}) = p$, $\Pr(E_n \to E_1) = p$ and $\Pr(E_i \to E_{i-1}) = q$, $\Pr(E_1 \to E_n) = q$.

i) Obtain the transitional probability matrix for this process

(2 marks)

ii) Check whether the chain is irreducible.

(2 marks)

iii) Suppose the chain is irreducible, classify the states in the chain

(7 marks)

b) At One-Berber Shop, customers arrive according to Poisson distribution with mean arrival rate of 5 per hour and the hair cutting is exponentially distributed with average haircut taking 10 minutes. It is assumed that because of his excellent reputation, customers are always willing to wait. Calculate the following

i) Average number of customers in the shop

(3 marks)

ii) Average number of customers waiting for a haircut.

(2 marks)

iii) The percent of time an arrival can walk right in without having to wait. (2 marks)

iv) The percent of customers who have to wait prior to getting in the Berber's chair (2 marks)

QUESTION FIVE (20 MARKS)

a) Define A birth process has the generating function $G(s,t) = se^{-\lambda t}\left[1-\left(1-e^{-\lambda t}\right)s\right]^{-1}$. Show that mean and variance of the process depends on t (7 marks)

b) Consider the process $X(t) = A\ Cos\ \lambda t + B\ Sin\ \lambda t$, where A and B are uncorrelated random variables with mean 0 and variance 1 and λ is a positive constant. Show that $X(t)$ is a weakly stationary process.

(5 marks)

c) An insecticide manufacturing firm is in the process of developing a mosquito insecticide. The firm has come up with 3 generics of such insecticide X, Y and Z, which it has been testing in Kisumu County over the last few months. A random survey of the changing patterns between the three brands for the month of April and May carried out by interviewing 530 households revealed the following;

	To	April	X	Y	Z	May
From	X	150	120	27	3	156
	Y	180	36	135	9	194
	Z	200	0	32	168	180

To keep track of the changing patterns, the firm has been interviewing the same households, none of which ceased to stay in Kisumu. Further the firm believes that all other Markov process conditions hold fairly well.

i) Determine the transition matrix for this problem (2 marks)

ii) Find the market share of each brand in the months of July (3 marks)

iii) Further, obtain the long-run distribution of the three insecticide brands (3 marks)