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KIRIRI WOMENS' UNIVERSITY OF SCIENCE AND TECHNOLOGY
UNIVERSITY EXAMINATION, 2023/2024 ACADEMIC YEAR
FOURTH YEAR, SECOND SEMESTER EXAMINATION
FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS
KMA 414: MEASURE THEORY AND PROBABILITY

Date:

Time:

INSTRUCTIONS TO CANDIDATES

ANSWER QUESTION ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a) Define the following terms;
   i) $\sigma-$ field (2 Marks)
   ii) Measurable function (2 Marks)
   iii) Power sets (2 Marks)
   iv) Measurable space (2 Marks)

b) Explain the condition under which a sequence of independent random variables said to be stable. (2 Marks)

c) Distinguish between;
   i) Finite measure and $\sigma-$ finite measure (2 Marks)
   ii) Generalized probability measure and conditional probability measure (2 Marks)

d) Let X be a random variable with Bernoulli distribution. Define a standardized version of X as S_n. State the central limit theorem using S_n. Hence or otherwise, give the pdf of S_n (5 Marks)

e) Show that Lebesque outer measure is translation invariant. (6 Marks)

f) Show that φ and $\Re$ are Lebesque measurable. (5 Marks)

QUESTION TWO (20 MARKS)

a) Two fair dice are tossed. Let S be the sum of the outcomes. Define X as follows;

$$X=\begin{cases} 0, & if\ S<6 \\ 1, & if\ 6\leq S<10 \\ 2, & Otherwise \end{cases}$$

i) Give the sample space for the experiment (2 Marks)

ii) Obtain the inverse of X and hence the $\sigma-¿$field induced by X (4 Marks)

iii) Give the pdf of X (2 Marks)

b) Explain the following probability concepts;

i) Classical concept (2 Marks)

ii) Limiting concept (3 Marks)

iii) Axiomatic concept (4 Marks)

c) Let $B \subset \Re$, where $\Re$ is the set of real numbers. Show that the inverse of I_A is a $\sigma-field$ (3 Marks)

QUESTION THREE (20 MARKS)

a) Let X be a random variable with a characteristic function $Q_x(t)$

i) Give the expression for $Q_x(t)$ in terms of trigonometric ratios. Identify the real and imaginary parts (2 Marks)

ii) Show that $Q_x(t)$ is continuous (4 Marks)

iii) Show that for any characteristic function $Q_x(t)$

$$4\,\mathrm{Re}[1-Q_x(t)] \geq \mathrm{Re}[1-Q_x(2t)]$$

(4 Marks)

b) Show that a $\sigma-¿$field is a monotone field and vice versa. (6 Marks)

c) Distinguish between monotone increasing and monotone decreasing sequences. Hence or otherwise determine convergence of the sequence below;

$$A_n = \left\{ w:\ 4-\frac{1}{n} < w < 7+\frac{3}{2n} \right\}$$

(4 Marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following;

i) Convergence in probability (3 Marks)

ii) Convergence in distribution (3 Marks)

b) Let g an even non-decreasing and non-negative Borel function between $[0, \infty)$. Then for a positive constant a, show that;

$$\frac{E[g(x)]-g(a)}{Sup\ g(x)} \leq \Pr[|X|>a] \leq \frac{E[g(x)]}{g(a)}$$

(10 Marks)

c) State and prove the mutual convergence theorem. (4 Marks)

QUESTION FIVE (20 MARKS)

a) Define stability of a random variable and use Chebychev's inequality to explain Bernoulli Weak Law of Large Numbers. Hence or otherwise give the conditions for Borel Strong Law of Large Numbers. (7 Marks)

b) Let $X_1, X_2, \ldots\ldots, X_n$ be independent Bernoulli random variables with parameter p. Define the central limit theorem. Hence or otherwise, give the Linderberg-Leivy form of central limit theorem. (4 Marks)

c) Define the term Lebesque outer measure. Hence or otherwise, state and prove the ideal properties of Lebesque outer measure. (9 Marks)