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SOME ASPECTS OF NUMERICAL RANGES OF BOUNDED LINEAR OPERATORS IN A COMPLEX HILBERT SPACE

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ABSTRACT

With mathematics analysts' interest shifting from finite-dimensional inner product spaces to infinite-dimensional Hilbert spaces and with consequent shift of interest from matrices to linear operators, their focus of attention changed from quadratic forms to numerical ranges of linear operators. In case of a bounded linear operator, the closure of the numerical range, apart from including the spectrum of the operator turns out to be a convex subset of the complex plane. It is this aspect that makes the study of the numerical range more appealing and worthy of the increasing attention currently directed towards it.

First, we give an alternative proof to the most important property of numerical range that for any bounded any linear operator, the numerical range is a convex set. Secondly, we show that for a hyponormal operator, the convex hull of the spectrum is the closure of numerical range. We also show that the same holds for a subnormal operator. Lastly, we prove that if the numerical range is closed, then every point λ in the boundary of the numerical range at which the boundary is not a differentiable arc is an eigenvalue for T.

Definitions And Consequences

We assume, unless otherwise mentioned, that H denotes a complex Hilbert space with the inner product function $\langle , \rangle : H \times H \to C$

Definition 1

Let $T \in B(H)$. The set $W(T)$ given by

$$W(T) = \{\langle Tx, x\rangle : x \in H \text{ and } ||x|| = 1\}$$

is called the numerical range of the operator T and the number sup $\{|\lambda| : \lambda \in W(T)\}$ is called the numerical radius of T, denotes by $W(T)$.

Theorem 1:

Let $(X, ||\cdot||)$ be a normed linear space and K be a convex subset of X. Then $\overline{K}$, the strong closure of K is also convex. [1]

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The most important property of $W(T)$ is given in the so-called Hausdorff-Toeplitz theorem.

Theorem 2:

For any $T \in B(H)$, $W(T)$ is a convex set. [2]

Remark: Many proofs of this theorem have been constructed in the literature of operator theory. We provide an alternative proof later.

It follows that the closure $\overline{W(T)}$ is also convex (see theorem 1)

Since $\overline{W(T)}$ is convex and contains $\sigma(T)$, it contains the convex hull of $\sigma(T)$.

Thus we have

Theorem 3:

Let $T \in B(H)$, then $conv\sigma(T) \subseteq \overline{W(T)}$. [3]

Now for any $T \in B(H)$ the spectrum $\sigma(T)$ is a compact subset of C. A nontrivial fact of finite-dimensional Euclidean geometry is that the convex hull of a compact set is closed.

The most useful formulation of this fact for the plane C is that the convex hull of a compact set is the intersection of all the closed half-planes that include it.

The question now arises: can the closure of the numerical range be very much larger than the spectrum? The answer is in the affirmative and the example exposed below demonstrates it to the extreme.

Example 1

Consider the Hilbert space $H = \forall^2$ of dimension 2 over C and take the orthonormal basis $\{e_1, e_2\}$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Define $T : H \to H$ linearly through $Te_1 = e_2$ and $Te_2 = \overline{0}$

Thus the matrix T with respect to the given orthonormal basis is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

0 is the only eigenvalue of T; thus $\sigma(T) = P_\sigma(T)$

(since H is finite dimensional) = $\{0\}$.

Let $x = (z_1, z_2) \in H$, so $x = z_1e_1 + z_2e_2$ and $Tx = z_1e_2 = (0, z_1)$

Consequently, $\langle Tx, x\rangle = \langle (0, z_1), (z_1, z_2)\rangle = z_1\overline{z}_2$

If $\|x\| = 1$, then $|z_1|^2 + |z_2|^2 = 1$. Thus

$$W(T) = \left\{z_1\overline{z}_2 : z_1, z_2 \in C \text{ and } |z_1|^2 + |z_2|^2 = 1\right\}$$

Putting $\lambda = z_1\bar{z}_2$, we have

$$|\lambda| = |z_1||z_2| = |z_1|\sqrt{1-|z_1|^2} \text{ ; hence}$$

$$W(T) = \{\lambda \in C : |\lambda|^2 = |z_1|^2 (1 - |z_1|^2), \text{ where } 0 \le |z_1| \le 1 \text{ and } z \in C\}$$

If $|z_1| = 0$ or 1, then $\lambda = 0$.

We find the maximum value of $|\lambda|$ as $|z_1|$ varies over the closed interval [0, 1]. For this, we can use the techniques of calculus, or the following simpler procedure:

$$|\lambda|^2 = |z_1|^2\left(1-|z_1|^2\right) = \left(|z_1|^2 - |z_1|^4\right) = \left\{\frac{1}{4} - \left(|z_1| - \frac{1}{2}\right)^2\right\}^2$$

Since $|\lambda| \ge 0$, we note that maximum value of $|\lambda|$ is $\frac{1}{2}$ and occurs when $|z_1| = \frac{1}{2}$

Hence

$$W(T) = \left\{\lambda \in C : |\lambda| \le \frac{1}{2}\right\} \tag{1}$$

Also $W(T) = \frac{1}{2}$, as is seen directly from (1).

Alternatively, we may observe that

$$|< Tx, x >| = |z_1\bar{z}_2| \le 1/2\left(|z_1|^2 + |z_2|^2\right) = \frac{1}{2}$$

For $z_1 = z_2 = \frac{1}{\sqrt{2}}$ we obtain $W(T) \ge \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. Hence $W(T) = \frac{1}{2}$.

Note that

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| : x \in H \text{ and } \|x\| = 1\} \\ &= \sup\{\|(0, z_1)\| : x = (z_1, z_2) \text{ and } \|x\| = 1\} = 1 \end{aligned}$$

Thus $W(T) = \frac{1}{2}\|T\|$ for this operator.

Theorem 4

If T is subnormal and S is its minimal normal extension, then $\sigma(S) \subset \sigma(T)$ [2]

Theorem 5

Let $T \in B(H)$ be hyponormal. Then

$$Conv\sigma(T) = \overline{W(T)} \tag{2}$$

Proof: In theorem 3, we have seen that for any $T \in B(H)$

$$Conv\sigma(T) \subseteq \overline{W(T)}$$

Therefore, when T is hyponormal, we need to show that

$$Conv\sigma(T) \supseteq \overline{W(T)} \tag{3}$$

This would follow if we show that for any closed half-plane which contains $\sigma(T)$ also contains $\overline{W(T)}$. Then (the statements made following theorem 3) the intersection of all the closed half-planes that include the (compact) set $\sigma(T)$ is the convex hull of $\sigma(T)$ and this intersection contains $\overline{W(T)}$.

By translation and rotation this reduces to showing that Re $\sigma(T) \leq 0$ implies $\text{Re}\overline{W(T)} \leq 0$

Let $\|\tilde{x}\| = 1$ and $T\tilde{x} = (a+ib)\tilde{x} + \tilde{y}$ with $a, b \in \boldsymbol{R}$ and $\tilde{x} \perp \tilde{y}$

For all $c > 0$, we note that $c \in \rho(T)$, and hence $(T - cI)^{-1}$ exists. Let $dist(c, \sigma(T)) = \tilde{c}$.

Then $\tilde{c} \geq c$, since $\text{Re}\sigma(T) \leq 0$. It follows that

$$\|(T - cI)^{-1}\| = (\tilde{c})^{-1} \leq c^{-1}$$

Thus $\|(T - cI)^{-1}z\|^2 \leq c^2 \|z\|^2$ for all $z \in H$

Replacing $(T - cI)^{-1}z$ by x and z by $(T - cI)x$, we get

$$c^2\|x\|^2 \leq \|(T - cI)x\|^2 \text{ for all } x \in H.$$

Put $x = \tilde{x}$. Then since $\|\tilde{x}\| = 1$

$$c^2 \leq \|(T - cI)\tilde{x}\|^2 \tag{4}$$

Now

$$\begin{aligned}\|(T-cI)\tilde{x}\|^2 &= \|(a+ib)\tilde{x}+\tilde{y}-c\tilde{x}\|^2 \\ &= \|[(a-c)+ib]\tilde{x}+\tilde{y}\|^2 \\ &= |(a-c)+ib|^2+\|\tilde{y}\|^2 \\ &= (a-c)^2+b^2+\|\tilde{y}\|^2\end{aligned}$$

Hence (4) yields

$$c^2 \le (a-c)^2+b^2+\|\tilde{y}\|^2 \text{ i.e., } 2ac \le a^2+b^2+\|\tilde{y}\|^2$$

Since this holds for all $c > 0$, $\mathrm{Re}\langle Tx, x\rangle = a \le 0$.

Since normal and subnormal operators are hyponormal, it follows that the relations:

$$r(T) = \|T\| \text{ and } Conv\sigma(T) = \overline{W(T)} \quad \text{(see [2])}$$

hold for normal and subnormal operators.

Referring back to theorem 4, we saw that if $T \in B(H)$ is subnormal and S is its minimal normal extension, then $\sigma(S) \subset \sigma(T)$.

Theorem 6

The closure of the numerical range of a subnormal operator is the convex hull of its spectrum.

Proof: If $T \in B(H)$ is subnormal and S is its minimal normal extension, then by theorem 4, $\sigma(S) \subset \sigma(T)$.

Since S is an extension of T, then

$$W(T) \subseteq W(S) \tag{5}$$

Since S is normal (and hence hyponormal), we have

$$\overline{W(S)} = Conv\sigma(S) \subseteq Conv\sigma(T) \subseteq \overline{W(T)} \subseteq \overline{W(S)}$$

Thus all the sets $\overline{W(S)}$, $Conv\sigma(S)$, $Conv\sigma(T)$, $\overline{W(T)}$ are the same. In particular

$$\overline{W(T)} = Conv\sigma(T)$$

Corollary 7

The closure of the numerical range of a subnormal operator is the same the closure of the numerical range of its minimal normal extension. [4].

Special Points On The Boundary Of A Numerical Range

Lemma 8

Let T be a linear operator on two dimensional Hilbert space H_2. If the matrix T (which is a 2×2 matrix) has distinct eigenvalues λ_1 and λ_2 and the corresponding eigenvectors x_1 and x_2, so normalized that $\|x_1\| = \|x_2\| = 1$, then $W(T)$ is a closed elliptical disc with foci at λ_1 and λ_2; if $r = |\langle x_1, x_2\rangle|$ and $\delta = \sqrt{1-r^2}$, then the minor axis is $r\frac{\lambda_1 - \lambda_2}{\delta}$ and the major axis is $\frac{\lambda_1 - \lambda_2}{\delta}$.

If T has only one eigenvalue λ, then $W(T)$ is the (circular) disc with center at λ, and radius $\frac{1}{2}\|T - \lambda I\|$.

From This Lemma, We Can Prove Theorem 2 In The Following Fashion;

If a and b are distinct points in $W(T)$, then there exists x and $y \in H$ such that

$$a = \langle Tx, x\rangle, b = \langle Ty, y\rangle, \|x\| = \|y\| = 1$$

Let M be the subspace $[\{x, y\}]$ spanned by x and y. Hence M is a closed linear space of H of dimension 2 over C:(Assume the contrary, that $\{x, y\}$ is linearly dependent over C. Then $x = \alpha y$ for some $\alpha \in C$ with $|\alpha| = 1$. We then have

$$\langle Tx, x\rangle = \langle T(\alpha y), \alpha y\rangle = |\alpha|^2 \langle Ty, y\rangle = \langle Ty, y\rangle, \text{ i.e., } a = b, \text{ a contradiction.}$$

Hence $\{x, y\}$ must be linearly independent over C.

Let $P_{x,y}$ be the orthogonal projector on H onto M.

Take a $z \in M$ with $\|z\| = 1$ .We have $P_{x,y} z = z$.

Thus $TP_{x,y} z = Tz$. Now Tz need not be in M.

However, $P_{x,y} Tz \in M$. Consequently, $P_{x,y} TP_{x,y} z = P_{x,y} Tz$.

Thus $\langle P_{x,y}T P_{x,y}z, z\rangle = \langle P_{x,y} Tz, z\rangle = \langle Tz, P_{x,y} z\rangle = \langle Tz, z\rangle$.

Now $\langle Tz, z\rangle \in W(T)$ and we thus obtain $W(P_{x,y} TP_{x,y}) \subset W(T)$.

Since $W(P_x TP_{y,x})$ is an elliptic (or circular) disc, it follows that $W(T)$ is convex..

We now prove:

Theorem 9

Let $T \in B(H)$ and $W(T)$ be a closed set. Every point λ in the boundary of $W(T)$ at which the boundary is not a differentiable arc is an eigenvalue for T.

Proof: It is well known that the boundary of $W(T)$ being a convex function, is differentiable, except perhaps at an at most countable set of points.

Let λ be a point of non-differentiability and x, $\|x\| = 1$ such that $\lambda = \langle Tx, x \rangle$.

Also, at λ there exists a left and right tangents such that the angle between these tangents is smaller than π. Let y be arbitrary in H and $P_{x,y}$ be the orthogonal projector on H onto the linear subspace $[\{x, y\}]$. The operator $T_1 = P_{x,y} TP_{x,y}$ has a closed elliptic disc as its numerical range, and since no circle contained in $W(T)$ can pass through λ, it follows that the ellipse $W(T_1)$ is a line segment or a point; thus λ is an eigenvalue with x as an eigenvector.

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