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Compound Joint-life Annuity Frailty Modeling

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Abstract. Grouping insureds in clusters such as joint-life annuities imposes statistical dependence. In this paper, we propose the shared compound frailty approach in collective valuation of joint-life annuity products where most applications have been in bio-statistics. The positive stable compound process used entails the frailty mixing distribution with the weighted exponential, generalized exponential and weighted Weibull as the base force of mortality distributions calibrated on a large Kenyan insurer joint-life last-survivor dataset. The findings shows that the positive stable generalized exponential model addresses time-varying heterogeneity effects positively and negatively associated with dependence.

Key words: Shared frailty, positive stable, generalized exponential, Bayesian inference, joint-life annuity.
AMS 2010 Mathematics Subject Classification Objects : 62P05.

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Résumé (Abstract in French) Le regroupement des assurés en grappes telles que les rentes viagères impose une dépendance statistique. Dans cet article, nous proposons l'approche de la fragilité composée partagée dans l'évaluation collective des produits de rente viagère dont la plupart des applications ont été faites en biostatistiques. Le processus composé stable positif utilisé implique la distribution de mélange frailty avec la Weibull exponentielle pondérée, la loi exponentielle généralisée et pondérée comme force de base des distributions de mortalité calibrées sur un grand ensemble de données d'un assureur kenyan sur la vie conjointe des derniers survivants. Les résultats montrent que le modèle exponentiel généralisé stable positif aborde les effets d'hétérogénéité variant dans le temps positivement et négativement associés à la dépendance.

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1. Introduction

Clayton (1978) proposed application of multivariate frailty model when two or more lives are not independent and assumed to share common risk U_i (frailty). A lot of research quantifying the effect of heterogeneity on annuity valuation apply uni-variate frailty models (Su and Sherris (2012); Gatzert *et al.*(2012); Onchere (2013); Fong (2015)). Other researchers have applied shared frailty dependence models but in medical fields see e.g., Hanagal (2020). Closest to this paper, Onchere *et al.*(2021a) have applied shared frailty in (national-level) insureds life table rates to account for dependence. In the present paper, the scientific interest is in advancing the collective valuation aspects and baseline modeling in Onchere *et al.*(2021a) by applying a real life (market-level) last-survivor dataset and considering the weighted exponential (WE), generalized exponential (GE) and weighted Weibull (WW) baselines respectively. We believe this advancements makes the model more flexible and acceptable. Valuation of annuity products is determined by the actuarial values (AVs). A suitable model for the mortality rates is needed when computing AVs to minimize the risk of biased valuation (Coppola *et al.*(2000);Luciano *et al.*(2016);Gildas *et al.*(2018);

Arias and Cirillo (2021);Onchere *et al.*(2021b)).

Uni-variate frailty models have been adopted by many authors, for instance, Gatzert *et al.*(2012) applies a frailty model to represent mortality heterogeneity in risk classes optimization for sub-standard annuities that incorporates reported risks. Olivieri and Pitacco (2016) suggests using frailty modeling to classify risk factors for life annuity portfolios. Specifically, the authors identify risk clusters in a population based on assigned frailty estimates for each cluster. Onchere *et al.*(2021b) suggests using the non-central gamma frailty model to account for unobserved heterogeneity in annuity pricing.

The paper is organized as follows. In Section two, we discuss the materials and methods applied in the research. Section three describes the Bayesian analysis used in parameter estimation, along with prior specifications, likelihood distribution of the model parameters and the Open Source Bayesian Inference using Gibbs Sampling (OpenBUGS) algorithms. In Section four, we present the model specifications and results. A brief discussion and summary is provided in Section five.

2. Materials and methods

2.1. Assumptions

(a) Conditional on the shared risk U_i, times-to-death of insureds in the i^{th} group are independent.

(b) The frailty U_i is assumed to have a multiplicative effect on the insureds hazard: $m_{ij}(t|u) = u_i m_0(t)$.

2.2. Data

We considered 398 joint-life last-survivor annuities data between calender years 2001-2018 from a large Kenyan insurance company. Demographic information of policyholders includes; gender, main life date of birth, spouse date of birth, effective date, main life term and spouse term. This dataset will be used in the baseline parameter estimation procedures and to compute the insureds level of association.

2.3. Joint-life last survivor annuity

The proposed methodology is applicable to any type of joint-life annuity contracts. Its application to joint-life last survivor annuities is discussed because it is issued more frequently in the market. Conceptually, this refers to a policy that commences payment as long as two annuitants are alive and continues for the entire life of the last survivor. The goal is to guarantee steady income upon attaining the retirement age; hence, annuities are comparable to single-life pensions. We can express this as a series of payments to annuitants (X,Y) of say, C annually beginning at time 1

whose AV is expressed as:

$$C \cdot a_{\overline{XY}} = C \cdot \sum_{t=1}^{\omega} v^t S_{\overline{XY}}(t). \tag{1}$$

Where the times-to-death random variable $T = Max(x, y)$, (x, y) represents the male (X) and female (Y) times-to-death, $v^t = (1 + i)^{-t}$ is the present value factor corresponding to interest rate i, $S_{\overline{XY}}(t)$ is the last survivor probability and $(\omega = 109)$ the limiting age.

Considering dependence the AV in Equation (1) becomes

$$C \cdot \sum_{t=1}^{\omega} v^t [S_X(t) + S_Y(t) - S_{XY}(t)], \tag{2}$$

under independence we have

$$C \cdot \sum_{t=1}^{\omega} v^t [S_X(t) + S_Y(t) - S_X(t) \cdot S_Y(t)], \tag{3}$$

where $S_X(t), S_Y(t), S_{XY}(t)$ are the survivor probabilities for the male (X), female (Y) and dependent lives (X, Y) respectively. Supposing an annuity series of payments of C annually, then by applying the equivalence principle we get net single premium of P:

$$P = C \times a_{\overline{XY}}. \tag{4}$$

Traditionally, due to simplicity in computations most insurers assume independence in valuation of joint-lives thereby adapting the AV shown in Equation (3). Frailty dependence modeling accounts for both heterogeneity and dependence thus adapting the AV shown in Equation (2).

2.4. Shared Frailty Approach

The mortality rate for a shared frailty approach is represented as:

$$m_{ij}(t|u_i) = u_i m_0(t); t > 0, \tag{5}$$

where $m_{ij}(t|u_i)$ *denotes the* j^{th} *individuals mortality rate in the* i^{th} *group,* u_i *is the shared risk and* $m_0(t)$ *the baseline hazard.* When we assume conditionally independent expected times-to-death for a specified shared risk, the bi-variate marginal survivor can be expressed as:

$$S_{XY}(x, y) = \mathbb{L}_{U_i}(M_0(x) + M_0(y)), \tag{6}$$

where $\mathbb{L}(.)$ is the Laplace transform and $M_0(.)$ the cumulative baseline mortality rate. (See proof in **Appendix A1**, page 3210).

2.5. The Positive Stable Mixture

The positive stable probability density function (pdf) is expressed as:

$$f(u) = -\frac{1}{\pi u}\sum_{j=1}^{\infty}\frac{\Gamma(j\alpha+1)}{j!}(-u^{-\alpha}k/\alpha)^j \sin(\alpha j\pi); k>0, u>0, 0<\alpha\leq 1.$$

Where k is the scale parameter and α the index parameter. To ensure identifiability let $k=\alpha$. (Proof found in Hougaard (2000)).

$$f(u) = -\frac{1}{\pi u}\sum_{j=1}^{\infty}\frac{\Gamma(j\alpha+1)}{j!}(-u^{-\alpha})^j \sin(\alpha j\pi); u>0, 0<\alpha\leq 1.$$

The Laplace is a unique type of Power-Variance-Function Laplace Onchere (2013) expressed as:

$$\mathbb{L}_U(s) = \exp\{-\frac{k}{\alpha}s^{\alpha}\}.$$

Where again to ensure the model is identifiable we set $k=\alpha$.

$$\mathbb{L}_U(s) = \exp\left(-s^{\alpha}\right), 0<\alpha\leq 1. \tag{7}$$

The suggested frailty mixture has many merits. Firstly, it is easily implementable due to its simplified Laplace derived in Equation (7). Secondly, the positive stable variance is infinite. As a result, more heterogeneity can be accounted for than when a frailty mixture is used with fixed variance.
The marginal bi-variate survivor function applying Equations (6) and Laplace (7) is:

$$S_{XY}(x,y) = \exp\{-(M_0(x)+M_0(y))^{\alpha}\}, \tag{8}$$

In dependence frailty modeling, the frailty distribution can be specified using the relative risk measure $A(x,y)$ seen as a local extension of Kendall's tau τ (Wienke (2011)). $A(x,y)$ describes how dependence of bi-variate mortality rates changes with time. The relative risk for the first-life, given an exposure of the second life to an event as opposed to being event-free is represented as:

$$A(x,y) = \frac{S_{XY}(x,y)\frac{\partial^2}{\partial x\partial y}S_{XY}(x,y)}{\frac{\partial}{\partial x}S_{XY}(x,y)\frac{\partial}{\partial y}S_{XY}(x,y)}. \tag{9}$$

$A(x,y)>1$ represents positive association, $A(x,y)<1$ represents negative association and $A(x,y)=1$ independence.
We present below two examples with specific frailty mixing densities to obtain the relative risk.

Example 1. Using the positive stable frailty mixing distribution and integrated base force of mortality $M_0(.)$ the relative risk $A(x,y)$ defined in Equation (9) is expressed as:

$$A(x,y) = 1-(1-\frac{1}{\alpha})(M_0(x)+M_0(y))^{-\alpha}. \tag{10}$$

(See proof in the **Appendix A2**, page 3210). This relative risk is time-varying and dependent on times-to-death. When α takes on values near zero high dependence is observed between X and Y, while α near one indicates low dependence. $\alpha = 1$ and $\alpha = 0$ corresponds to maximal independence.

Example 2. Using the Gamma frailty mixing distribution and integrated base force of mortality $M_0(.)$, the relative risk $A(x, y)$ is expressed as:

$$A(x, y) = (1 + \sigma^2), \tag{11}$$

where σ^2 is the frailty variance. (See proof in the **Appendix A3**, page 3210). This relative risk is constant and independent of times-to-death.

2.6. Baseline Hazards

For annuity valuation purposes parametric base force of mortality estimation is desired (see Frees *et al.*(1996)). The GE, WE and WW distributions are suggested as an improvement of the Weibull, lognormal and gamma baselines applied in Onchere *et al.*(2021a) since these give more flexibility in modeling as shown below.

2.6.1. Generalized Exponential

If $m_0(t)$ follows the GE with pdf $f_0(t) = \varpi\rho(1-\exp(-\varpi t))^{\rho-1}\exp(-\varpi t); t > 0, \varpi, \rho > 0$. Where ϖ is a scale parameter and ρ the shape parameter. Then the survival, hazard and cumulative hazard functions are respectively;

$$\begin{aligned}
S_0(t) &= 1 - [1 - \exp(-\varpi t)]^{\rho}, \\
m_0(t) &= \frac{\rho\varpi(1-\exp(-\varpi t))^{\rho-1}\exp(-\varpi t)}{1-[1-\exp(-\varpi t)]^{\rho}}, \\
M_0(t) &= -\ln(1-[1-\exp(-\varpi t)]^{\rho}).
\end{aligned}$$

The GE hazard is increasing ($\rho > 1$), decreasing ($\rho < 1$) or constant ($\rho = 1$).

The log-likelihood function $\ell(\varpi, \rho)$ considering a given set of times-to-death data $\underline{t} = (t_1, t_2, ..., t_k)$ is given by

$$\ell(\varpi, \rho|\underline{t}) = k\log(\varpi\rho) + (\rho - 1)\sum_{i=1}^{k}\log(1-\exp(-\varpi t_i)) - \varpi\sum_{i=1}^{k} t_i \tag{12}$$

The estimates $\hat{\varpi}, \hat{\rho}$ can be derived from the non-linear equations $\frac{\partial\ell}{\partial\varpi} = 0$ and $\frac{\partial\ell}{\partial\rho} = 0$ using any iterative methods. In the current paper we apply OpenBUGS algorithms.

2.6.2. Weighted Exponential

If $m_0(t)$ follows the WE with pdf $f_0(t) = (1 - \exp(-a\lambda t))\frac{1+\lambda}{\lambda} a \exp(-at); t > 0, a, \lambda > 0$. Where a is a scale parameter and λ the shape parameter (Gupta and Kundu (2009)). Then the survival, hazard and cumulative hazard functions are respectively;

$$
\begin{aligned}
S_0(t) &= \frac{1+\lambda}{\lambda}[\exp(-at) - \frac{\exp\{-(1+\lambda)ax\}}{1+\lambda}],\\
m_0(t) &= \frac{(1-\exp(-a\lambda t))a\exp(-at)}{\exp(-at) - \frac{\exp\{-(1+\lambda)ax\}}{1+\lambda}},\\
M_0(t) &= -\ln(\frac{1+\lambda}{\lambda}[\exp(-at) - \frac{\exp\{-(1+\lambda)ax\}}{1+\lambda}]).
\end{aligned}
$$

The log-likelihood $\ell(a, \lambda)$ considering a given set of times-to-death data $\underline{t} = (t_1, t_2, ..., t_k)$ is expressed as:

$$\ell(a, \lambda|\underline{t}) = k\log(\frac{a}{\lambda}(1+\lambda)) + \sum_{i=1}^{k}\log(1-\exp(-a\lambda t_i)) - a\sum_{i=1}^{k} t_i \tag{13}$$

The estimates $\hat{a}, \hat{\lambda}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial a} = 0$ and $\frac{\partial \ell}{\partial \lambda} = 0$.

2.6.3. Weighted Weibull

If $m_0(t)$ follows the WW with pdf $f_0(t) = abx^{b-1}\frac{1+\lambda^b}{\lambda^b}\exp\{-ax^b\}(1-\exp\{-a\lambda x^b\})$. Where λ is a scale parameter and a, b are shape parameters.(Roman, R. (2010)) Then the survival, hazard and cumulative hazard functions are respectively;

$$
\begin{aligned}
S_0(t) &= \frac{1+\lambda^b}{\lambda^b}[\exp\{-ax^b\} - \frac{1}{1+\lambda^b}\exp\{-ax^b(1+\lambda^b)\}],\\
m_0(t) &= \frac{abx^{b-1}\exp\{-ax^b\}(1-\exp\{-a\lambda x^b\})}{\exp\{-ax^b\} - \frac{1}{1+\lambda^b}\exp\{-ax^b(1+\lambda^b)\}},\\
M_0(t) &= -\ln(\frac{1+\lambda^b}{\lambda^b}[\exp\{-ax^b\} - \frac{1}{1+\lambda^b}\exp\{-ax^b(1+\lambda^b)\}]).
\end{aligned}
$$

The log-likelihood $\ell(a, b, \lambda)$ considering a given set of times-to-death data is given by

$$\ell(a, b, \lambda|\underline{t}) = k\log(\frac{ab(1+\lambda^b)}{\lambda^b}) - a\sum_{i=1}^{k} t_i^b + (b-1)\sum_{i=1}^{k}\log t_i + \sum_{i=1}^{k}\log(1-\exp(-a\lambda {t_i}^b)). \tag{14}$$

The estimates $\hat{a}, \hat{b}, \hat{\lambda}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial a} = 0, \frac{\partial \ell}{\partial b} = 0$ and $\frac{\partial \ell}{\partial \lambda} = 0$

3. Bayesian analysis

In the Bayes technique, any unknown parameter is regarded as a varying quantity and its distribution is derived from what is known about them. This technique is used as estimation procedures in actuarial studies e.g. by Scollnik (1993) in analyzing concurrent mathematical statements for assurance pricing also by Rosenberg and Young (1999) in studying time-varying dependence when there exists shifts in variance estimation. The Bayes specification technique is executed in the following procedure using OpenBUGS. Firstly, we defined log-likelihood functions as shown in Equations 12,13 and 14 respectively. Due to the lack of earlier knowledge about the base force of mortality specifications, non-informative priors are selected and presumed to be flat. That is, $Gamma(0.001, 0.001)$ for the positive specifications see Hanagal (2020).

Actual data for males and female times-to-death is obtained from the large Kenyan insurer last-survivor dataset. Model specifications will be obtained by considering the life terms from 39 exact through to 68 exact as given in the real dataset. Burn-in period is fixed at 30000 as per the Brooks-Gelman-Rubin (BGR) Figure 1 to ensure posterior distributions sequences of draws have low auto-correlation and is obtained from the values of a run of Markov chain (Brooks and Gelman (1998)). Thereby diminishing the effects of the initial density. We simulate 2 chains in parallel and thereafter stationarity will be monitored upon completion of 100000 replications. If convergence is achieved the mean of the posterior distribution is selected as a point estimate. Low Akaike Information Criteria (AIC), Deviance Information Criteria (DIC) and Bayesian Information Criteria (BIC) would indicate a better model.

1. *DIC*=$\bar{A} + pA$ where $\bar{A}$ is the posterior mean of $-2 \times \log \mathbf{L}$ indicating the goodness-of-fit quality of the proposed methodology to the dataset. $\hat{A} = -2 \times \log \mathbf{L}$ is the posterior mean of stochastic nodes and $pA = \bar{A} - \hat{A}$ measures the ultimate parameters specifications (see Spiegelhalter *et al.*(2002)).

2. *AIC*=$\hat{A} + 2\rho$ where; ρ is the aggregate specifications.

3. *BIC*=$\hat{A} + \rho \times \log(m)$ where; m is the sample size. The BIC is useful as it considers the BIC penalty for all parameters being estimated. OpenBUGS algorithm applied to analyse the GE model is shown in the **Appendix B**, page 3211.

3.1. BGR Diagnostics and Trace Graphs

The diagnostic graphs for BGR nodes convergence examined is illustrated in Figure 1. As the simulation chains continues, the total-sequence simulated value (green curvature) and average within-sequence intervals (blue curvature) are examined. Their ratio (red curvature) is observed to merge to one after 30000 simulations hence providing a good burn-in period.

Table 1: Base Force of Mortality Parameter Estimates.

Baseline	**Parameters**	**AIC**	**DIC**	**BIC**
1. GE	$\rho_1 = 7050; \varpi_1 = 0.1816$	2483	2477	2491
	$\rho_2 = 4514; \varpi_2 = 0.1579$	2604	2602	2612
2. WE	$a_1 = 0.03896; l_1 = 0.006283$	3626	3623	3627
	$a_2 = 0.03571; l_2 = 0.009485$	3699	3697	3700
3. WW	$a_1 = 6.302E-6; b_1 = 3.074; l_1 = 2.642$	3037	3019	3036
	$a_2 = 1.49E-5; b_2 = 2.881; l_2 = 3.151$	3283	3038	3282

3.2. Discussion

On the basis of Bayes inference based on Gibbs sampling the GE density is selected as the AIC, DIC and BIC values is lower in comparison with the other distributions. The model specifications applied in this study is displayed in Table 1 upon implementing the Bayes technique discussed previously.

3.3. Goodness-of-fit test

Comparing Kaplan-Meier survival function plot (black curve) for the real dataset verses the GE model survivor functions for males and females (red curves) we visually observe a good fit (see Figure 3).

Further, a chi-square and Kolmogorov-Smirnov goodness-of-fit test of the dataset for GE survivor rates is displayed in Table 2. As shown, the chi-square and Kolmogorov-Smirnov test p-values are $\geq$ 5%. Thus, we do not reject the null hypothesis that the Kenyan last-survivor rates can be effectively modeled using a GE survivor function at 5% significance level. The GE quantile-quantile (Q-Q) graph in Figure 2 displays a straight line through a majority of the quantiles further justifying the GE as a better fit. We can thus conclude that the GE best fits the data.

Table 2: Chi-square and Kolmogorov-Smirnov Goodness-of-fit of GE to the Kenyan Last-survivor Rates.

Name	Value (Males data)	Value (Females data)
Chi-squared statistic	812	1056
Degree of freedom	784	1024
Chi-squared p-value	0.2371	0.2374
Kolmogorov-Smirnov test statistic	0.19231	0.2222
Kolmogorov-Smirnov p-value	0.6069	0.3047

4. Model Specifications and Results

The positive stable GE frailty bi-variate survivor function is described explicitly as

$$S(x,y) = \exp\{-(-\ln(1-[1-\exp(-\varpi_1 x)]^{\rho_1}) - \ln(1-[1-\exp(-\varpi_2 y)]^{\rho_2}))^{\alpha}\}. \quad (15)$$

The positive stable GE dependence mixture is displayed in Figure 4 where (x,y) represents the male and female annuitants times-to-death. The baseline hazard parameters are $\rho_1 = 7050; \varpi_1 = 0.1816; \rho_2 = 4514; \varpi_2 = 0.1579$ computed from the Bayes inference technique.

The joint last-survivor local measure of association is determined from the large Kenyan insurer joint-life last survivor data-set. We consider 398 joint-life annuitants data in-force between 2001-2018. The dependence $S_{XY}(x,y)$ survivor rates is computed using Kendall's tau ($\tau = 0.7357$) obtained from the Kenyan joint lives dataset. Here $\alpha = 0.2643$ obtained using the relation $\alpha = 1 - \tau$.

The independence survivor rates $S_X(x) \cdot S_Y(y)$ is computed from Equation 15 when $\alpha = 1$. The AVs and net single premiums are generated as previously discussed. Considering a case where the annuitants expect to receive Ksh 200000 annually.

4.1. Impact on survivor rates

As shown in Figure 4 the survivor function under independence is higher initially compared to the dependence assumption. This is explained by downside impacts of association incorporated in the frailty methodology (e.g., occurrence of a contagious disease or an accident). Thus incorporating short-term association that exists. Afterwards, there is an underestimation of survival rates in the independence approach in comparison with the dependence approach because of longevity risk. I.e the longer the joint lives survive beyond a certain time, the better their chances of survival are. In this case, the long-term association is catered for. Therefore the independence approach under-values the survival risk at extreme advanced ages.

4.2. Impact on annuity net single premiums

Moreover, in Figure 5 when the annuity net single premiums are compared at 7% interest rate (central bank of Kenya interest rate as at May 2022) it is observed that the independence approach leads to over-valuation of the insurance firm's product at the start of the policy and under-valuation later because mortality increase decelerates at extreme old ages. This can be explained by the fact that the insurer offers high prices when the survivor rates are high and vice versa because the benefits are paid for the entire life of the last survivor.

5. Conclusion

The present paper scientific interest is in advancing the collective valuation aspects and baseline modeling in Onchere *et al.*(2021a) by applying a real life

(market-level) last survivor dataset and considering the WE, WW and GE baselines respectively. Specifically, Onchere *et al.*(2021a) applied national-level life table rates and the Weibull baseline distribution. The findings arrived at is that the GE baseline distribution provides a good fit to the Kenyan last-survivor dataset compared to the other baseline distributions following the models comparison criteria. Further, applying the positive stable GE frailty approach demonstrates that the relative risk is time-varying and dependent of lifetimes when compared to the independence approach.

The shared frailty shows a decrease in the expected obligation of the insurance firm at early annuitants ages (due to low survivor rates) but an increase in liability at extreme old ages (due to high survivor rates) when association is considered. A good explanation for this trend is that the survivor rates for frail couples is assumed to be low in the initial stages of the policy, later increase in survivor rates at very old ages since high-risk couples have already died, emphasizing the importance of dependence modeling in collective valuation of annuity contracts. Thus assumptions of joint-life independence can result in biased annuity valuation.
Future research involves extensions to advanced compound frailty processes. In-order to account for negative association, for instance, death of one couple leading to a positive effect on survival of the other exists. We hope to explore this point in a future paper.

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5.1. Appendix

Appendix A1. The bi-variate conditional survivor function for a given shared frailty U_i is:

$$S_{XY}(x,y|u_i) = S_X(x|u_i) \cdot S_Y(y|u_i),$$

applying

$$S(t|u) = \exp\left(-\int_0^t m_0(x|u)dx\right) = \exp\left(-uM_0(t)\right),$$

leads to

$$S_{XY}(x,y|u_i) = \exp\left(-u_i[M_0(x) + M_0(y)]\right).$$

Using expectation

$$S_{XY}(x,y) = E[\exp\left(-u_i[M_0(x) + M_0(y)]\right)] = \mathbb{L}_{U_i}(M_0(x) + M_0(y)).$$

Appendix A2. From Equation (9) and the bi-variate survivor in [8] we have

$$\frac{\partial}{\partial x}S_{XY}(x,y) = -\alpha(M_0(x) + M_0(y))^{\alpha-1}m_0(x)\exp\{-(M_0(x) + M_0(y))^{\alpha}\},$$

$$\frac{\partial}{\partial y}S_{XY}(x,y) = -\alpha(M_0(x) + M_0(y))^{\alpha-1}m_0(y)\exp\{-(M_0(x) + M_0(y))^{\alpha}\},$$

$$\frac{\partial^2}{\partial_x \partial y}S_{XY}(x,y) = [\alpha^2(M_0(x) + M_0(y))^{2\alpha-2} - \alpha(\alpha-1)(M_0(x) + M_0(y))^{\alpha-2}]$$

$$\cdot m_0(x)m_0(y)\exp\{-(M_0(x) + M_0(y))^{\alpha}\}.$$

$$A(x,y) = \frac{[\alpha^2(M_0(x) + M_0(y))^{2\alpha-2} - \alpha(\alpha-1)(M_0(x) + M_0(y))^{\alpha-2}]}{-\alpha(M_0(x) + M_0(y))^{\alpha-1}m_0(x)\exp\{-(M_0(x) + M_0(y))^{\alpha}\}}$$

$$\cdot\frac{m_0(x)m_0(y)\exp\{-(M_0(x) + M_0(y))^{\alpha}\}\exp\{-(M_0(x) + M_0(y))^{\alpha}\}}{-\alpha(M_0(x) + M_0(y))^{\alpha-1}m_0(y)\exp\{-(M_0(x) + M_0(y))^{\alpha}\}},$$

$$A(x,y) = 1 - \frac{\alpha(\alpha-1)(M_0(x) + M_0(y))^{\alpha-2}}{\alpha^2(M_0(x) + M_0(y))^{2\alpha-2}},$$

$$A(x,y) = 1 - (1 - \frac{1}{\alpha})(M_0(x) + M_0(y))^{-\alpha}.$$

Appendix A3. Using the identifiable gamma Laplace $\mathbb{L}_{U_i}(s) = (1 + s\sigma^2)^{-1/\sigma^2}$ and bi-variate survivor $S_{XY}(x,y) = \mathbb{L}_{U_i}(M_0(x) + M_0(y)) = (1 + \sigma^2(M_0(x) + M_0(y)))^{-1/\sigma^2}$ we have From Equation (9):

$$\frac{\partial}{\partial x}S_{XY}(x,y) = m_0(x)(1 + \sigma^2(M_0(x) + M_0(y)))^{-1/\sigma^2-1}$$

$$\frac{\partial}{\partial y}S_{XY}(x,y) = m_0(y)(1 + \sigma^2(M_0(x) + M_0(y)))^{-1/\sigma^2-1}$$

$$\frac{\partial^2}{\partial x \partial y} S_{XY}(x,y) = \frac{m_0(x)m_0(y)(1+\sigma^2)}{(1+\sigma^2(M_0(x)+M_0(y)))^{1/\sigma^2+2}}$$

$$A(x,y) = \frac{(1+\sigma^2(M_0(x)+M_0(y)))^{-1/\sigma^2} \cdot \frac{m_0(x)m_0(y)(1+\sigma^2)}{(1+\sigma^2(M_0(x)+M_0(y)))^{1/\sigma^2+2}}}{m_0(x)(1+\sigma^2(M_0(x)+M_0(y)))^{-1/\sigma^2-1} \cdot m_0(y)(1+\sigma^2(M_0(x)+M_0(y)))^{-1/\sigma^2-1}}$$

$$A(x,y) = (1+\sigma^2)$$

Appendix B: R-Program Code.

GE OpenBUGS R-code

```
MODEL = function()\{
        for(i in 1:398)\{
                dummy[i]=0
                dummy[i] $\sim$ dloglik(logLike[i])
                logLike[i] = log(b*a)+(b-1)*log(1-exp(-a*s[i]))-a*s[i]
        \}
        a $\sim$ dgamma(0.001,0.001)
        b $\sim$ dgamma(0.001,0.001)
\}

write.model(MODEL,"MODEL.txt")

INIT=function() \{
        list(a=dgamma(0.001,0.001),b=dgamma(0.001,0.001))
\}

DATA=list(s=X-lifetime)\#for the females 's=Y-lifetime'
BUGS=bugs(data=DATA, inits=INIT, parameters.to.save=c("b","a"),
model.file="MODEL.txt",n.chains=2,n.iter=100000,n.burnin=30000,
codaPkg=TRUE, debug=T)
```

Fig. 1: BGR Diagnostics and Trace Graphs for $GE(\rho, \varpi)$ where $b = \rho, a = \varpi$.

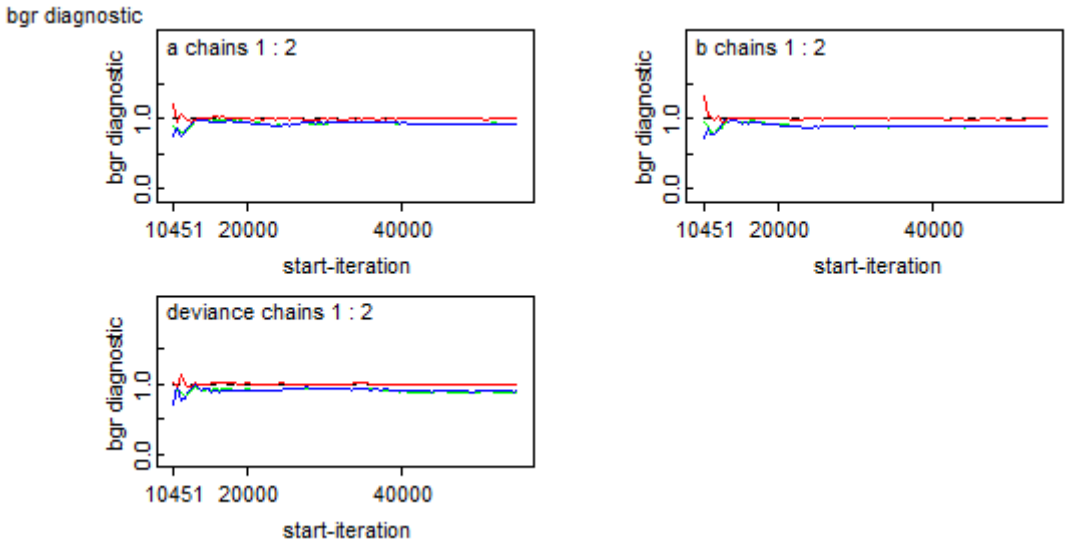


Fig. 2: Q-Q Plot for GE model to the Kenyan last-survivor rates.

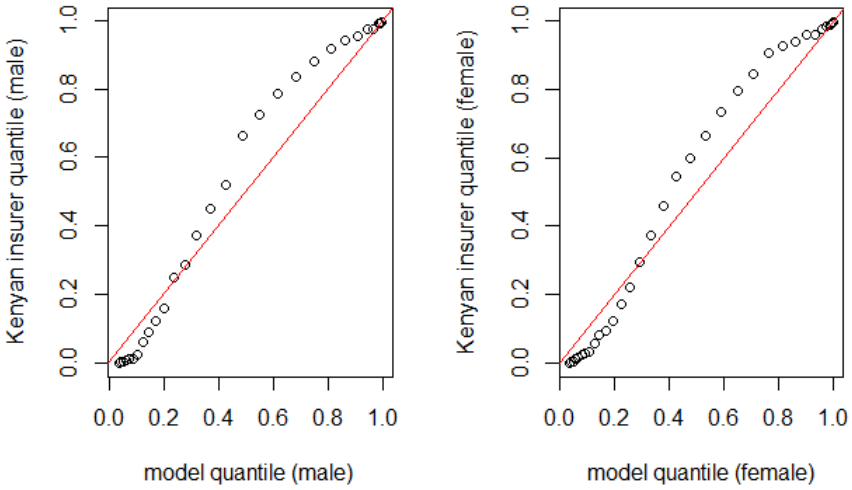


Fig. 3: Kaplan-Meier (black curve) versus GE (red curve) survivor curves.

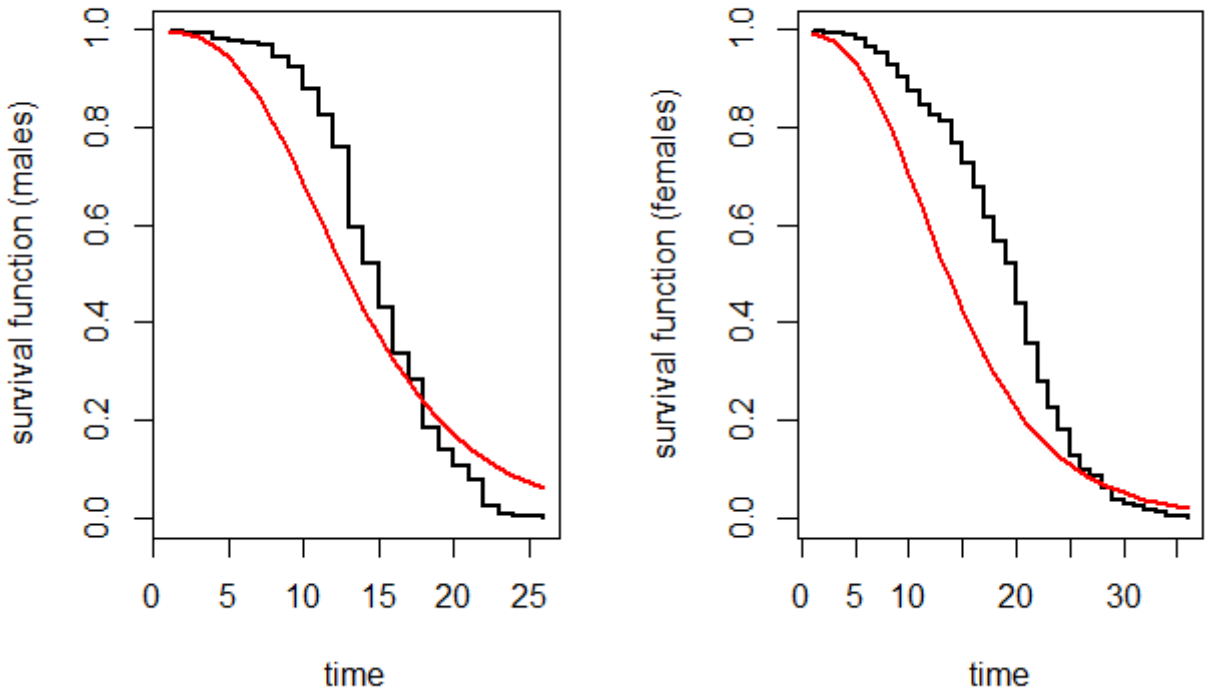


Fig. 4: Dependence (red curve) versus Independence (black curve) Survival Rates

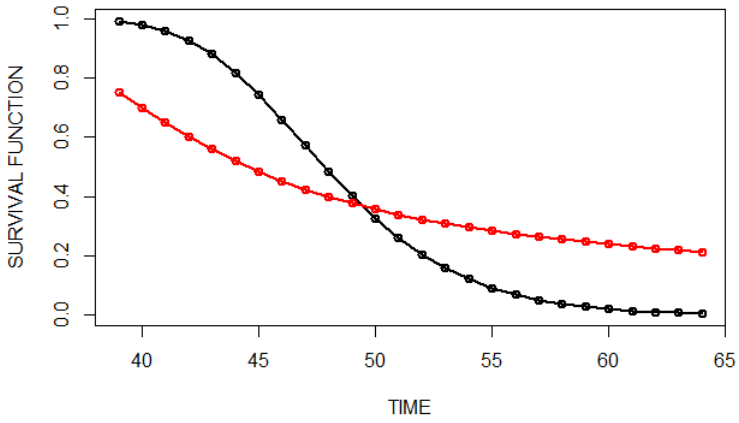
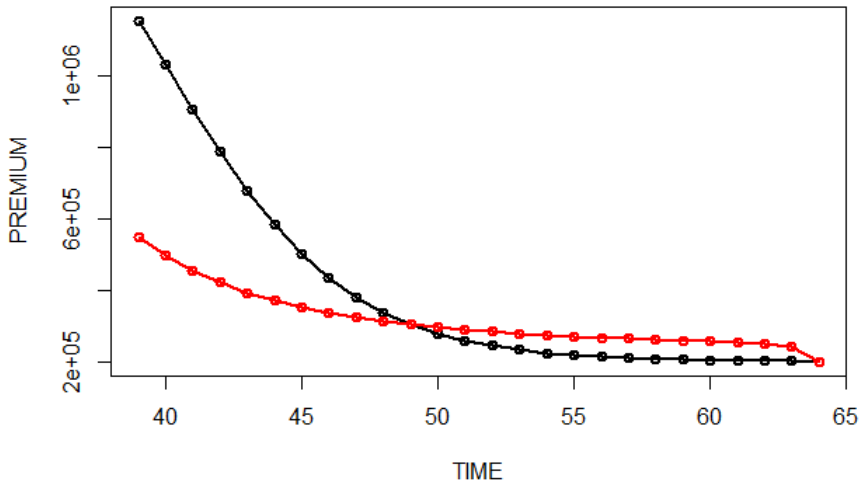


Fig. 5: Dependence (red curve) versus Independence (black curve) Net Single Premium Rates



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