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Graduation of term assurance data using frailty approach

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Abstract. Frailty models have been used in literature to account for heterogeneity among insureds in-terms of mortality. In this article, we compare the gamma and the non-central gamma as frailty distributions with the exponentiated exponential and exponentiated Weibull as baseline hazards. We adopt a fully Bayesian approach to calibrate the baselines based on crude mortality rates from a major Kenyan insurer. Comparing the gamma-exponentiated Weibull with the non-central gamma-exponentiated Weibull models shows that the non-central gamma provides a good fit to the real life data-set and is therefore recommended for valuation.

Key words: frailty; exponentiated exponential; exponentiated Weibull; Bayesian inference; term assurance; gamma-exponentiated Weibull; non-central-gamma-exponentiated Weibull.
AMS 2010 **Mathematics Subject Classification Objects :** 62*P*05.

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Résumé (French Abstract). Des modèles *frailty* ont été utilisés dans la littérature pour prendre en compte de l'hétérogénéité des assurés en termes de mortalité. Dans cet article, nous comparons le modeles gamma et le gamma non-central en tant que distributions de *frailty* avec une loi exponentielle et Weibull exponentiées comme fonctions de hazard. Nous adoptons une approche entièrement bayésienne pour calibrer les niveaux de référence en fonction des taux de mortalité bruts d'une grande compagnier kenyane. La comparaison du Weibull gamma-exponentié gamma avec les modèles Weibull non-central-gamma exponentaited montre que le gamma non-central fournit un bon ajustement à l'ensemble de données de la vie réelle et est donc recommandé pour l'évaluation.

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1. Introduction and motivation

Heterogeneity mortality modeling is an important aspect of risk management in actuarial work. Since the pioneering work of [Cox (1972)] on regression modeling accounting for the effects of reported risk factors, [Vaupel *et al.*(1979)] extended the work to accounts for both the reported risk factors *e.g.* smoking habits having an impact on mortality and unreported risk factors *e.g.* genetic factors affecting mortality in a frailty framework. There has since been growing literature on frailty heterogeneity modeling in actuarial work *e.g.* [Wang *et al.*(1998)] apply a Gamma-Gompertz frailty model for projection of human mortality improvements. The authors graduate the reported mortality improvement factors in a published Society of Actuaries tables. [Meyricke and Sherris (2013)] have adopted the frailty model to quantify the impact of heterogeneity due to underwriting factors and frailty on annuity values. The results showed that heterogeneity remains after underwriting and frailty significantly impacts the fair value of both standard and underwritten annuities. [Pitacco (2018)] applies frailty modeling to analyze the impact of frailty on the results of cash flows and profits of life assurance and life annuity portfolios. [Onchere *et al.*(2021)] applies the non-central gamma-Weibull frailty mixture to model heterogeneity in insurance pricing. The findings shows that ignoring heterogeneity due to other factors affecting mortality other than age and sex only leads to an underestimation of the mortality rates. In literature frailty is commonly

modeled with a gamma distribution because of mathematical convenience, but no biological reasons. In this article, we compare the gamma frailty mixture with the non-central gamma (*NCG* henceforth) mixture applied in [Onchere *et al.*(2021)] with the aim of improving the baseline modeling. In particular, the exponentiated exponential and exponentiated Weibull baselines are considered.

2. Materials and methods

2.1. Term Assurance

The pricing of assurance products is largely influenced by the choice of the mortality (μ_x) model ([Stehno *et al.*(2010)];[Gildas *et al.*(2018)];[Onchere *et al.*(2021)]). A term assurance is an insurance contract that pays a specified amount say, B on the death of the policyholder within a specified term say, k years. The expected present value (*EPV*) of this benefit is given by

$$B \int_0^k v^t \; s_x(t) \mu_{x+t} \; dt, \tag{1}$$

where x is the age of the insured life, v^t is the present value factor and $s_x(t)$ the survival probability.

2.2. Frailty Model

[Vaupel *et al.*(1979)] described the frailty model as

$$\mu(x|\omega) = \omega \; \mu_0(x), \tag{2}$$

where $\mu_0(x)$ is the population's base force of mortality. The non-negative variable ω entails other factors affecting mortality other than age. Strong (weak) individuals are associated with low (high) values of ω.

Definition 1. Let X be the future lifetime random variable with a continuous distribution. A non-negative random variable ω is called frailty if the conditional hazard function is given by

$$\mu(x|\omega) = \omega \; \mu_0(x), \; x > 0,$$

where $\mu_0(x)$ is the baseline age-specific hazard function.

Proposition 1. *The conditional survival function is given by*

$$S(x|\omega) = \exp\left(-\omega H_0(x)\right), \; x > 0, \tag{3}$$

where

$$H_0(x) = \int_0^x \mu_0(t)\ dt$$

is the cumulative baseline hazard.

Proof of Proposition 1. The conditional survival function $S(x|\omega)$ is given by

$$S(x|\omega) = \exp\left(-\int_0^x \mu(t|\omega)\ dt\right)$$

and by using **Definition 1**, we have

$$S(x|\omega) = \exp\left(-\int_0^x \omega\ \mu_0(t)\ dt\right)$$

which simplifies to

$$S(x|\omega) = \exp\left(-\omega H_0(x)\right),\ x > 0.$$

Proposition 2. *The Uni-variate marginal survival function is given by*

$$S(x) = L_\omega(H_0(x)),\ x > 0, \tag{4}$$

where $L(s)$ is the Laplace transform.

Proof of Proposition 2. The marginal survival function $S(x)$ is given by

$$S(x) = \int_0^\infty S(x|\omega)\ f(\omega)\ d\omega.$$

By using expectation

$$S(x) = \mathbb{E}[S(x|\omega)]$$

and from Proposition 1, we have

$$S(x) = \mathbb{E}\left[\exp\left(-\omega H_0(x)\right)\right],$$

which simplifies to

$$S(x) = L_\omega(H_0(x)),\ x > 0.$$

2.3. Proposed Frailty Distributions

2.3.1. Gamma frailty distribution

Let ω be gamma distributed with shape parameter p and scale parameter φ. The probability density function (*pdf*) is given by

$$f(\omega) = \frac{\varphi^p \omega^{p-1} \exp(-\varphi\omega)}{\Gamma(p)};\ \omega > 0,\ p > 0,\ \varphi > 0. \tag{5}$$

The Laplace transform allowing for identifiability, *i.e.*, $\mathbb{E}(\omega) = 1$ is given by

$$L(s) = \left(1 + s\sigma^2\right)^{-1/\sigma^2}.$$

The marginal survival, density and hazard functions are respectively:

$$S(x) = \left(1 + \sigma^2 H_0(x)\right)^{-1/\sigma^2}, \tag{6}$$

$$f(x) = \frac{\mu_0(x)}{(1 + \sigma^2 H_0(x))^{(1+1/\sigma^2)}}, \tag{7}$$

$$\mu(x) = \frac{\mu_0(x)}{1 + \sigma^2 H_0(x)}. \tag{8}$$

2.3.2. Non-central gamma frailty distribution

The *pdf* for the *NCG* distribution with Y being a mixing of the distributions of

$$\omega_1 + \omega_2 + \cdots + \omega_N$$

with respective weights

$$\frac{\exp(-\lambda)\lambda^n}{n!},$$

where

$$\omega \sim \Gamma(b, \lambda), \quad N \sim \mathcal{P}(a\lambda)$$

leads to the convolution

$$f(\omega, a, b, \lambda) = \sum_{n=0}^{\infty} \frac{\exp(-a\lambda)(a\lambda)^n}{n!} \left[\frac{\omega^{b+n-1} \exp(-\frac{\omega}{\lambda})}{\Gamma(b+n)\lambda^{b+n}}\right]. \tag{9}$$

Proposition 3. *Given that*

$$Y = \omega_1 + \omega_2 + \cdots + \omega_N,$$

with respective weights

$$\frac{\exp(-\lambda)(\lambda)^n}{n!},$$

where

$$\omega \sim \Gamma(b, \lambda),\ N \sim \mathcal{P}(a\lambda).$$

Then

$$f(\omega, a, b, \lambda) = \sum_{n=0}^{\infty} \frac{\exp(-a\lambda)(a\lambda)^n}{n!} \left[\frac{\omega^{b+n-1}\exp\{-\frac{\omega}{\lambda}\}}{\Gamma(b+n)\lambda^{b+n}}\right].$$

proof of Proposition 3.

$$\begin{aligned}\mathbb{P}(Y=n) &= \sum_{n=0}^{\infty} \mathbb{P}(\omega_1 + \omega_2 + \cdots + \omega_N | N=n)\ \mathbb{P}(N=n) \\ &= \sum_{n=0}^{\infty} \left[\frac{\omega^{b-1}\exp(-\omega/\lambda)}{\Gamma(b)\lambda^b}\right]^{*n} \frac{\exp(-a\lambda)(a\lambda)^n}{n!},\end{aligned}$$

where

$$\left[\frac{\omega^{b-1}\exp(-\omega/\lambda)}{\Gamma(b)\lambda^b}\right]^{*n}$$

is the $n-th$ fold convolution power of

$$\left[\frac{\omega^{b-1}\exp(-\omega/\lambda)}{\Gamma(b)\lambda^b}\right]$$

and $\mathbb{P}$ is the probability of the associated event.

$$\mathbb{P}(Y=n) = \sum_{n=0}^{\infty} \left[\frac{\omega^{b+n-1}\exp(-\omega/\lambda)}{\Gamma(b+n)\lambda^{b+n}}\right] \frac{\exp(-a\lambda)(a\lambda)^n}{n!}.$$

The Laplace transform for the *NCG* distribution allowing for identifiability is given by

$$L_{\omega}(s) = \exp\left(-\frac{s}{1+0.5\sigma^2 s}\right). \tag{10}$$

[See Onchere *et al.*(2021) for *Proof.*]

The marginal survival, density and hazard functions are respectively:

$$S(x) = \exp\{-H_0(x)(1+0.5\sigma^2 H_0(x))^{-1}\},$$

$$f(x) = \mu_0(x)(1+0.5\sigma^2 H_0(x))^{-2}\exp\{-H_0(x)(1+0.5\sigma^2 H_0(x))^{-1}\}$$

and

$$\mu(x) = \mu_0(x)(1+0.5\sigma^2 H_0(x))^{-2}. \tag{11}$$

We present below two examples with specific baseline distributions to find the frailty hazard functions with explicit expressions.

Example 1.

If $\mu_0(x)$ follows an exponentiated exponential (*EE*) distribution with *pdf*

$$f_0(x) = \gamma\alpha(1-\exp(-\gamma x))^{\alpha-1}\exp(-\gamma x);\ x>0,\ \gamma>0,\ \alpha>0.$$

Then the survival, hazard and cumulative hazard functions are respectively:

$$S_0(x) = 1-[1-\exp(-\gamma x)]^{\alpha},$$

$$\mu_0(x) = \frac{\alpha\gamma(1-\exp(-\gamma x))^{\alpha-1}\exp(-\gamma x)}{1-[1-\exp(-\gamma x)]^{\alpha}},$$

and

$$H_0(x) = -\ln(1-[1-\exp(-\gamma x)]^{\alpha}).$$

From Equation (8), the gamma-*EE* frailty hazard is described explicitly as

$$\mu(x) = \frac{\alpha\gamma\left(1-\exp(-\gamma x)\right)^{\alpha-1}\exp(-\gamma x)}{1-[1-\exp(-\gamma x)]^{\alpha}}\left(1-\sigma^2\ln\left(1-[1-\exp(-\gamma x)]^{\alpha}\right)\right)^{-1}. \tag{12}$$

From Equation (11) the *NCG-EE* frailty hazard is described explicitly as

$$\mu(x) = \frac{\alpha\gamma\left(1-\exp\left(-\gamma x\right)\right)^{\alpha-1}\exp\left(-\gamma x\right)}{1-\left[1-\exp\left(-\gamma x\right)\right]^{\alpha}}\left(1-0.5\sigma^2\ln\left(1-\left[1-\exp\left(-\gamma x\right)\right]^{\alpha}\right)\right)^{-2}. \quad (13)$$

The *EE* hazard is increasing ($\alpha > 1$), decreasing ($\alpha < 1$) or constant ($\alpha = 1$).

Example 2.

If $\mu_0(x)$ follows an exponentiated Weibull (*EW*) distribution with *pdf*

$$f_0(x) = \alpha\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)^{\alpha-1}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right);\ x>0,\ \gamma>0,\ \alpha>0,\ \lambda>0.$$

Then the survival, hazard and cumulative hazard functions are respectively:

$$S_0(x) = 1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha},$$

$$\mu_0(x) = \frac{\alpha\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)^{\alpha-1}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)}{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha}},$$

and

$$H_0(x) = -\ln\left(1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha}\right).$$

From Equation (8) the gamma-*EW* frailty hazard is described explicitly as

$$\mu(x) = \frac{\alpha\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)^{\alpha-1}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)}{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha}}\left(1-\sigma^2\ln\left(1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha}\right)\right)^{-1}. \quad (14)$$

From Equation (11) the *NCG-EW* frailty hazard is described explicitly as

$$\mu(x) = \frac{\alpha(1-\exp\left(-\lambda x^{\gamma}\right))^{\alpha-1}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)}{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha}}\cdot\left(1-0.5\sigma^2\ln(1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\alpha})\right)^{-2}. \quad (15)$$

The *EW* has a positive support and is also a good choice for the baseline hazard. The hazard curve is monotone increasing if ($\gamma \geq 1$) and ($\alpha\gamma \geq 1$); monotone decreasing if ($\gamma \leq 1$) and ($\alpha\gamma \leq 1$); unimodal if ($\gamma < 1$) and ($\alpha\gamma > 1$) and bathtub shaped if ($\gamma > 1$) and ($\alpha\gamma < 1$).

2.4. Parameter Estimation

Bayesian inference is a modern statistical technique that accounts for uncertainty associated with the model parameters in the form of prior distributions. This method have been applied as estimation procedures in actuarial modeling *e.g.* by [Scollnik (1993)] in analysis of a simultaneous equations for insurance rate making and by [Rosenberg and Young (1999)] to analyze time-varying dependent data with possible variance shifts. In shared frailty models by [Hanagal (2020)] to assess unreported heterogeneity in individual risks to kidney infection.

In [Butt and Haberman (2004)] an assurance application of the frailty-based survival model is proposed. The authors discuss various choices and fit some models to assurance mortality data. The results obtained suggest a potential range of $\sigma^2 \approx (2.916, 14.444)$ in an insured population with $\sigma^2 = 14\%$ for the heterogeneous case. Given the outcomes of the investigation by [Butt and Haberman (2004)], in this research we consider $\sigma^2 = 14\%$ for the heterogeneous case.

The Bayesian parameter estimation strategy is implemented in the following algorithm run in open source Bayesian inference using Gibbs sampling (*OpenBUGS*). First, the proposal distributions for the likelihood is specified as *EE* (α, γ) and *EW* $(\alpha, \lambda, \gamma)$ respectively. Since we do not have prior information about baseline parameters non-informative prior distribution is picked and assumed to be flat. I.e gamma distributed random variables with mean 1 and variance 10000 for the positive parameters values. Similar approach is found in [Scollnik (1993)]; [Hanagal (2020)].

The hyperparameters initial values is chosen to be $\Gamma(0.0001, 0.0001)$. The actual data to be estimated by the model is specified to be the crude mortality rates obtained from real life mortality data-set. Parameters are estimated considering only the range of ages $[24, 65]$ as obtained from the real data-set. Burn in period is set at 30000 as per the Brooks-Gelman-Rubin (*BGR*) plot to ensure sequence of draws from the posterior distribution have minimal auto-correlation and can be found by taking values from a single run of the Markov chain. This diminishes the effect of the starting distribution. We run 3 chains in parallel and after $100,000$ iterations convergence will be monitored and if stationarity has been achieved (implying estimates are not dependent on the prior distributions) the mean posterior distribution will be picked as a point estimate. Models with smaller values of the deviance information criteria (*DIC*) values are preferred.

The *OpenBUGS* codes used to analyze the dataset using the *EE* and *EW* is available upon request.

Brooks-Gelman-Rubin Diagnostic and Trace Plots.

The *BGR* convergence diagnostic plots for the monitored nodes are presented in Figure 1. As the Markov chain Monte Carlo (*MCMC*) simulation progresses, the val-

BASELINE MODEL	PARAMETER ESTIMATES	DIC
1. Exponentiated Exponential	$\alpha = 0.7244, \gamma = 5.996$	-84.24
2. Exponentiated Weibull	$\alpha = 2.29, \gamma = 0.7357, \lambda = 5.683$	-104.4

Table 1: Parameter estimates for the baseline distributions.

ues of the total-sequence (green curve) and mean within-sequence interval width (blue curve) estimates are monitored. Their ratio (red curve) is seen to converge to one beyond 30000 iterations hence a probable choice for the burn-in period. The dynamic trace plots also monitored in Figure 1 is shown to be mean-reverting and the chains appear to mix freely implying stationarity has been achieved.

2.5. Model selection criteria

In the Bayesian framework we apply the *DIC* in model selection. ***DIC=***$\bar{D} + pD$ where $\bar{D}$ is the posterior mean of $-2logL$ measuring the quality of the goodness-of-fit of the considered model to the data. $\hat{D} = -2LogL$ is the posterior mean of stochastic nodes and $pD = \bar{D} - \hat{D}$ is the effective number of parameter. Smaller values of *DIC* indicates better models and could give negative values.

The parameter estimates for the baseline distributions are shown in Table 1. The *EW* would fit better compared to the *EE* since the *DIC* is smallest.

3. Data analysis and interpretation

3.1. The Data

To preserve confidentiality, we took a sub-sample of 732 term assurance policies that were in force in $2010 - 2015$ from a large Kenyan insurer. The data contains the policyholders details such as, date of birth, date when the contract started and the date of death. This data will be used to estimate the real hazard rate in ages $24 - 65$ as experienced by the policyholders.

3.2. Analysis and Interpretation

The aims of this exercise are:

(i) Firstly, is to show that when the gamma is applied as a frailty distribution the hazard rates are overestimated at all ages compared to the *NCG*.

(ii) Secondly, is to show the relevance of the *NCG* frailty mixture to graduate the insurer's crude mortality rates.

Assumptions:

Name	Value
Chi-squared statistic	1722
Degree of freedom	1681
Chi-squared p-value	0.2379

Table 2: Chi-squared goodness of fit of *EE* to the crude mortality rates

Name	p-value	Test Statistic
Kolmogorov-Smirnov test	0.03517	0.30952

Table 3: Goodness of fit using *K-S* test.

(a) The force of mortality, μ, is assumed piece-wise constant, taking a common value across each whole year of age $[x, x+1)$ similar assumption found in [Dodd *et al.*(2018)].

(b) The frailty model considered here is one without reported covariates since only survival data is available for analysis.

(c) The age at which life assurance policyholders buy term assurance cover is assumed to be between $24-65$ as given in the real data-set.

The Gamma-*EW* frailty and *NCG-EW* frailty models given in equations [14 and 15] respectively are as shown in Figure 2 where x is the future lifetime, $\sigma^2 = 0.14$, $\alpha = 2.29$, $\gamma = 0.7357$, $\lambda = 5.683$; $\mu_0(x) \sim EW(2.29, 0.7357, 5.683)$.

In Figure 2 graduation is done using the Gamma-*EW* (blue curve) and *NCG-EW* (red curve) frailty model both calibrated on the real term assurance mortality rates. This is compared with the real term assurance mortality rates (black curve). As shown the Gamma-*EW* overestimates the hazard rate at all ages compared to the *NCG-EW* model. The *NCG-W* provides a good fit to the actual claims experience data. The chi-square test Table 2 and Kolmogorov-Smirnov (*KS*) hypothesis test Table 3 for overall goodness of fit is significant for the model. The chi-squared goodness of fit test has p-value greater than 0.01, indicating that the distribution is a good fit. Similarly, the *KS* goodness of fit test has p-value greater than 0.01, indicating that the distribution is a good fit.

4. Discussions and Conclusion

In this article, we compare the gamma frailty mixture with the non-central gamma mixture applied in [Onchere *et al.*(2021)] with the aim of improving the baseline modeling. In particular, the exponentiated exponential and exponentiated Weibull baselines are considered. Both models are applied to real term assurance mortality data for comparison purposes. Using Bayesian inference the *EW* turns out to give a better fit since the *DIC* is smallest compared to the *EE*. As shown in Figure 2

the Gamma-*EW* model overestimates the hazard rates at all ages compared to the *NCG-EW* model since the hazard curve shifts upwards. The *NCG-EW* fits well to the insurers claims experience as shown in the chi-squared goodness of fit test Table 2 and *KS* hypothesis test Table 3 that is significant. The conclusion arrived at is that using the gamma as the frailty distribution may lead to inappropriate term assurance valuations resulting in high prices that negatively impacts marketability of term contracts. The gamma frailty index is time invariant and frailty remains constant throughout life. The *NCG* stochastic process represents time-varying frailty and is recommended for better term assurance valuations.

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Fig. 1: *BGR* Diagnostic Plot consistent with convergence and Dynamic Trace Plots for $EW(\alpha, \lambda, \gamma)$ where $a = \alpha, l = \lambda, r = \gamma$

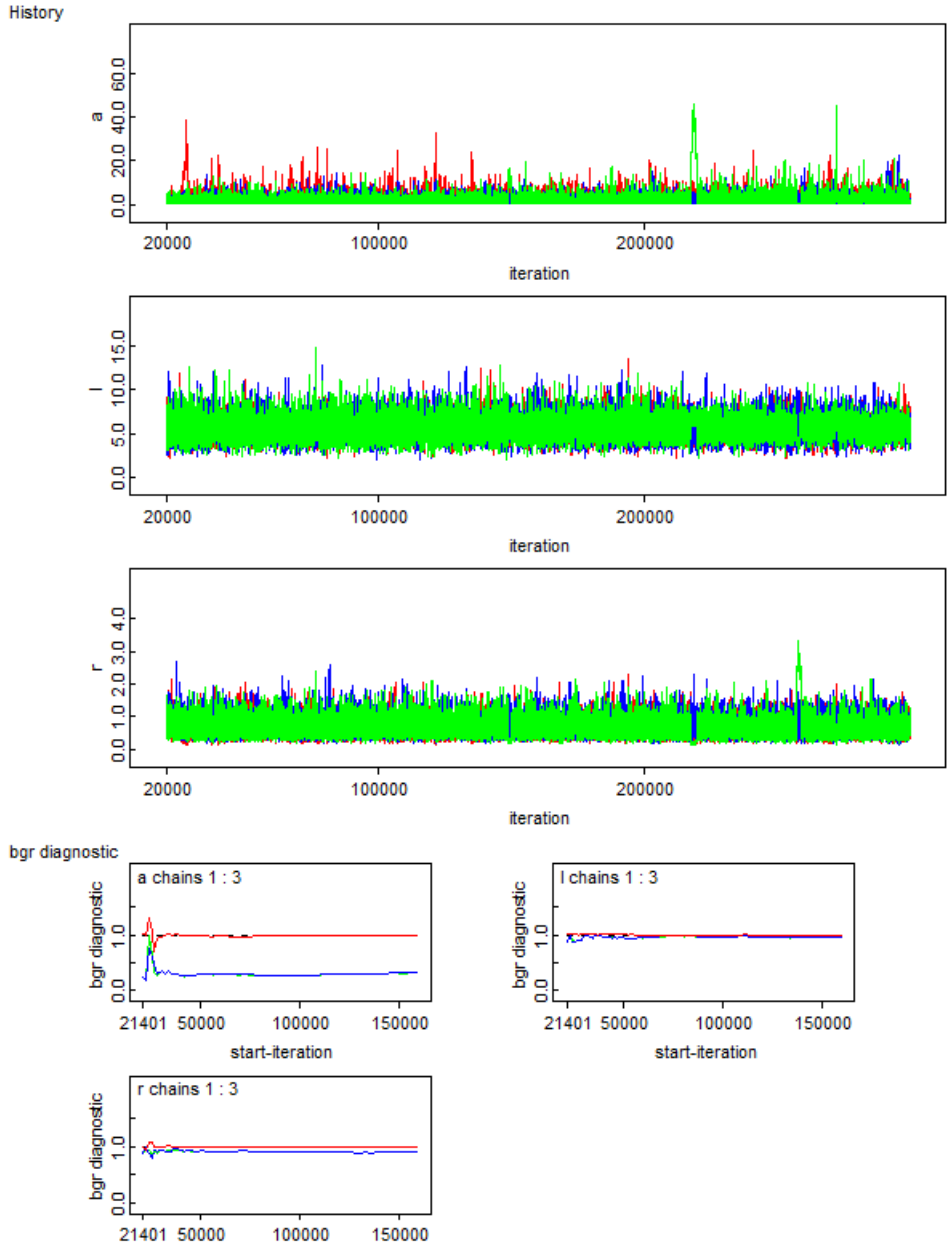

Fig. 2: Crude mortality rates and frailty hazard functions