A Note on Invariant subspaces of some operators in Hilbert Space.

Irene M. Muteti, Bernard M .Nzimbi, Stanley K. Imagiri and Jairus M. Khalagai

University of Nairobi,

School of Mathematics,

P.O Box 30197-00100,

Nairobi, Kenya.

Email address: irenemumb@yahoo.com

Abstract

In this paper, we show that if M is a nontrivial invariant for both T and S, then M is $ST-$invariant or $TS-$ invariant. An example is provided to illustrate that if M is $TS-$invariant, then it is not necessarily invariant for either T and S. However if TS, S and T have same structure and M is invariant for $TS,$ then it is also invariant for T and S.

Keywords : Invariant subspaces, Nilpotent operators

1. Introduction

The invariant subspaces of an operator plays a central role in operator they as they give information on the structure of the operator. They are a direct analogue of the eigen-vectors of a linear operator. The motivation behind the study of invariant subspaces come from the interest of structure of operators and from approximation theory. Let H be a Hilbert space and $B(H)$ denotes all bounded linear operators on H. A subspace M of H is a invariant under operator T if $T(M) \subseteq M$, that is, $x \in M$ for every $Tx \in M$ or $TM \subset M.$ If T is any subset of $B(H),$ we denote by $\{T\}'$ the commutant of T, that is $\{T\}' = \{T \in B(H): ST = TS\}.$

A subspace $M \subset H$ is said to be nontrivial hyper-invariant subspace (n.h.s) for a fixed operator in $T \in B(H)$ if 0$\neq M \neq H$ and $SM \subset M$ for each $S \in \{T\}'.$ An operator $T \in B(H)$ is nilpotent if $T^n = 0.$

Theorem 1.1 If $T \in B(H),$then the following subspaces are invariant under T:

(i) $\{0\}$.
(ii) H.
(iii) $Ker\,(T)$
(iv) $Ran\,(T)$

Proof. *(i)* If $x \in \{0\}$, then $x = 0$ hence $Tx = 0 \in \{0\}$. Thus {0} is invariant under T.

(ii) If $x \in H$, then $Tx \in H$ since T on Hilbert space H is bounded, then it is bounded below and hence its range is closed. Thus H is invariant under T.

(iii) If $x \in Ker(T), then\, Tx = 0$ and hence $Tx \in Ker\,(T)$. Thus $Ker\,(T)$ is invariant under T.

(iv) Note that, since the operators T on a Hilbert space H is bounded below and hence its range is closed subspace of H. Thus $T(Ran(T$)) is contained in $Ran(T)$. Let $x \in Ran(T)$, then $Tx \in Ran(T)$. Thus $Ran(T)$ is invariant under T.

Lemma 1.2 Let $U_1, U_2 \subset H$ be invariant subspaces. Then $U_1 \cap U_2$ and $U_1 + U_2$ are invariant subspaces.

Proof. Suppose U_1 and U_2 are both under T, and $\mathrm{u} \in U_1 \cap U_2$. Since U_1 is invariant under T, then $\mathrm{T}(\mathrm{u}) \in \mathrm{U}_1$. Similarly, since U_2 is invariant under T, then $\mathrm{T}(\mathrm{u}) \in \mathrm{U}_2$ and so

$\mathrm{T(u)} \in \mathrm{U}_1 \cap \mathrm{U}_2$. Thus $\mathrm{U}_1 \cap \mathrm{U}_2$ is invariant under T.

Suppose $u \in U_1 + U_2$. Then $u = u_1 + u_2$ where $u_i \in U_i$ for $i = 1, 2$. Applying the linear operator on both sides of the equation we have

$T(u) = T(u_1 + u_2) = T(u_1) + T(u_2)$.

Because U_1, U_2 are all invariant subspace under T, and since $u_i \in U_i$ we have T(u_i) $\in U_i$

For $i = 1{,}2$. Hence T (u) is contained in $U_1 + U_2$ and therefore $U_1 + U_2$ is invariant under T.

Proposition 1.3 Let T and L be nonzero on a Hilbert space H. If $LT = 0$, then $Ker(L)$ and $Ran\,(T)$ are nontrivial invariant subspaces for both T and L.

Proof. If LT=0, then Ran (T) $\subseteq$ Ker (L). Hence $T(\,Ker(L))$ $\subseteq$T (H) $= Ran\,(T) \subseteq Ker\,(L)$. Since $T \neq 0, Ran\,(T) \neq 0, so\ that\ Ker(L) \neq 0. S\,ince\ L \neq 0\ \ Ker(L) \neq H.$ Therefore $Ker\,(L)$ is nontrivial invariant subspace for T. Dually since T*L*=0, L*$\neq 0$ it follows that $Ker\,(T^*)^\perp$ is nontrivial invariant subspace for L^*, and hence $Ran(T) = Ker\,(T^*)^\perp$ is a nontrivial invariant subspace for L.

Remark 1.1 $Ker(L)$ and $\overline{Ran(T)}$ are invariant subspaces for L and T.

Proposition 1.3 leads to the following result.

Corollary 1.1 Every nilpotent operator has a nontrivial invariant subspace.

Proof. Recall that, an operator is nilpotent if $T^n = 0$. Thus $T^n = T(T^{n-1})$ which can be written as a product of two operators and by Proposition 1.3 Ker (T) and $\overline{Ran(T^{n-1})}$ are nontrivial invariant subspaces.

Proposition 1.4 Let $T \in B(H)$and M be subspace of a Hilbert space H. If M is $T -$invariant, Then $(T|_M)^* = PT^*|_M$ where P is the orthogonal projection of H onto M.

Proof. Let M be an invariant subspace for T so that $T\,(M) \subseteq M$, and let P be the orthogonal projection onto M. Since $P\,v = v$ for every $v \in M$ *and* using the fact that P is self-adjoint, we have

$\langle (T|_M)^* u, v\rangle = \langle u, T|_M v\rangle = \langle u, Tv\rangle = \langle u, TPv\rangle = \langle PT^* u, v\rangle = \langle PT^*|_M u, v\rangle$ for every

$u, v \in M,$ hence $(T|_M)^* = PT^*|_M$.

Proposition 1.5 Let $T,\ S \in B(H)$ and M be a nontrivial invariant subspace for both T and S. Then M is $TS -$ invariant.

Proof. If M is invariant for both T and S then we have $T(M) \subseteq M$ and $S(M) \subseteq M$.Thus we have $TSM = T(SM) \subseteq T(\,M) \subseteq M$. Therefore M is $TS -$ invariant.

Proposition1.6 Let $T, S \in B(H)$ and M be a nontrivial invariant subspace for both T and S. Then M is $ST -$invariant

Proof. If M is invariant for both T and S, then we have $T\,(M) \subseteq M$ and $S\,(M) \subseteq M.$ Thus we have $STM = S(TM) \subseteq S(M) \subseteq M$. Therefore M is $ST -$ invariant.

Question. If M is $TS -$invariant, is it true that M is $T -$invariant or $S -$ invariant?

Answer. We answer this question with the following example.

Let $TS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We observe that $Lat\,(TS) = \left\{\{0\}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, R^2\right\}$. However TS can be written, not uniquely, as a product of $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We notice that $M = span \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant for TS but it is not invariant for T and S.

This leads to the following remarks:

Remark 1.2 Let M be subspace of a Hilbert space H and $T, S \in B(H)$. If M is $TS-$ invariant, then M is not necessarily $T-$ or $S-$invariant.

However if TS, T and S have the same structure, then if M is $TS-$ invariant the M is also invariant for both T and S.

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