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Fitting a recursive model for the Beta II-Binomial mixture into other recursive models. A case of Binomial-Generalized Gamma distribution

Otwande Andrea[1*], Okuto Erick[1] and Ottieno Joseph[2]

[1]School of Mathematics and Actuarial Science
Jaramogi Oginga Odinga University of science and Technology.
P.O Box 210-40601, Bondo,Kenya.
[2]School of Mathematics and Actuarial Science,
University of Nairobi.
P.O Box 30197-01000, Nairobi,Kenya.
[*]Corresponding Author; Otwande Andrea e-mail: otwandeandrea@gmail:com

ABSTRACT: Probability distributions can be derived using recursive relations in probability. Various methods have been applied to attain this objective with Kotz(1965),Panjer(1981),Hesselager(1992) and Wang among others, leading in this study. Among the distributions obtained include mixed distributions of Poisson distribution. This paper however focuses on the mixture of beta-binomial which has had little focus as far as obtaining their recursive relations are concerned. The paper discusses its suitability and how it fits two recursive models of Panjer-Willmot and Hessselager's.

Key words: Beta-binomial, , differential equation, Hesselager Mixtures ,probability distribution,Panjer-Willmot, recursive model.

1.0 Introduction

Probability distributions can be constructed by using what is referred to as Pearson difference equations which are recursive relations in probabilities. These recursive models can be obtained through integration by parts and/or by applying ratio method. They can as well be obtained using iteration methods and through use of probability generating function techniques. Others can also be obtained and expressed in terms of confluent and Gauss hypergeometric functions. Various methods for developing recursive models for compounds distributions have been discussed.

Kotz(1965) considered the differential equation of $\frac{f(x+1)}{f(x)} = \frac{\alpha + \beta x}{1+\alpha} \qquad x = 0,1,2...$ which can be re arranged as

$(1+\alpha) f(x+1) = (\alpha + \beta x) f(x) \qquad x = 0,1,2...$. He used the model to generate several probability distributions by considering various cases of parameters α and β.He obtained Poisson distribution, negative binomial distribution, binomial distributions. Panjer (1981) obtained the recursive model $f(x) = (\alpha + \frac{\beta}{x}) f(x-1)$ where $f(x) = (X = x) \qquad x = 0,1,2....$ He obtained Poisson distribution, binomial distribution, negative binomial distribution, geometric distributions from their recursive models using probability generating function techniques. Sundt and Jewell (1981) used Panjer's recursive class of order one given as $(1+\alpha)f(x)=(\alpha+\beta x)f(x-1)$ to obtain zero-truncated, Poisson distribution, shifted geometric distribution, logarithmic distribution, zero-truncated negative binomial distribution using iteration techniques. Panjer-Willmot (1982) obtained recursive equation $[\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots+\alpha_k x^k]f(x)=[\beta_0+\beta_1(x-1)+\beta_2(x-1)^2+\ldots+\beta_k x^k]f(x-1)$ and used it to fit recursive models of various mixed distributions such as beta-geometric distribution, negative hypergeometric distribution, beta-negative binomial distributions. Their corresponding differential equations were also obtained. Hesselager (1992) obtained the recursive relation $f(x)\sum_{t=0}^{k} a_t n^t = f(x-1)\sum_{t=0}^{k} b_t n^t \qquad x=1,2,\ldots$ The recursive model obtained fitted beta-geometric distribution, hypergeometric distribution, negative hyper-geometric distribution, negative-binomial distributions as mixed distributions. Wang (1994) obtained a recursive model which was an extension of Hesselager's recursive model given as $f(x)\sum_{i=0}^{k} b_i n^i = \sum_{j=1}^{s}\left[f(x-1)\sum_{i=1}^{k} a_{ji}(x-j)^i\right]$ for $x=c,c+1,c+2..$ where c is positive integer and $f(x)=0$ for $x<c$. Willmot (1993) derived various mixed Poisson distributions and obtained their recursive relations which were fitted into Wang's recursive model. These included Sichel distribution, Poisson-beta distribution, Poisson-Generalized Pareto distribution and Poisson-Inverse Gamma distribution. From the above literature it is noted that various recursive models have been derived from mixed distributions using various techniques but with little focus on Beta-binomial mixtures.

This paper focuses on obtaining a recursive model from a mixture of binomial and generalized Beta II distribution. Specifically Binomial-Generalized Gamma mixed distribution. The recursive model obtained will be fitted into the recursive models for both Panjer-Willmot (1982) class and Hesselager's (1992) class. Their corresponding differential equations will also be obtained. The first section of this paper discusses on the formulation of the Binomial-Generalized Gamma mixture and its recursive relations. The second part of this section focusses on both Panjer-Willmot and Hesselager's recursive models formulation. Section two will focus on the results obtained when the model obtained is fitted into both Panjer-Willmot and Hesselager's recursive models. Section three discusses the results and lastly section four is about the conclusions and recommendations.

2.0 Method

2.1 Generalized Gamma Distribution

This is a special case of MacDonald's four parameter generalized distribution of the second kind. We wish to show that the generalized four parameter beta distribution of the second kind approaches the generalized Gamma distribution as $b\to\infty$ i.e., to find

$$\lim_{b\to\infty} g(\theta)=\lim_{b\to\infty}\frac{|C|\,\theta^{ca-1}}{d^{ca}B(a,b)(1+\left(\frac{\theta}{d}\right)^c)^{a+b}}$$

$$\text{Let } d=\beta b^{\frac{1}{c}}$$

$$g(\theta) = \frac{|C|\,\theta^{ca-1}}{\left(\beta b^{\frac{1}{c}}\right)^{ca} B(a,b)\left[1+\left(\frac{\theta}{\beta b^{\frac{1}{c}}}\right)\right]^{a+b}}$$

$$= \frac{|C|\,\theta^{ca-1}}{\beta^{ca} b^{a} B(a,b)\left[1+\frac{\theta^{c}}{\beta^{c} b}\right]^{a+b}}$$

$$= \frac{|C|\left(\frac{\theta}{\beta}\right)^{ca-1}}{\beta b^{a} B(a,b)\left[1+\frac{1}{b}\left(\frac{\theta}{\beta}\right)^{c}\right]^{a+b}}$$

$$= \frac{|C|\left(\frac{\theta}{\beta}\right)^{ca-1}\Gamma a+b}{\beta b^{a}\Gamma a\Gamma b\left[1+\frac{1}{b}\left(\frac{\theta}{\beta}\right)^{c}\right]^{a+b}}$$

$$= \frac{|C|\left(\frac{\theta}{\beta}\right)^{ca-1}}{\beta\Gamma a}\left(\frac{\Gamma a+b}{b^{a}\Gamma b}\right)\left[\frac{1}{1+\frac{1}{b}\left(\frac{\theta}{\beta}\right)^{c}}\right]^{a+b} \qquad (1)$$

For large values of b, the Gamma function can be approximated by Stirling formula.

$$\Gamma x \sim \sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}$$

$$\lim_{b\to\infty}\frac{\Gamma a+b}{b^{a}\Gamma b} \sim \frac{\sqrt{2\pi}e^{-(a+b)}(a+b)^{a+b-\frac{1}{2}}}{b^{a}\sqrt{2\pi}e^{-b}b^{b-\frac{1}{2}}}$$

$$= \frac{e^{-a}\ (a+b)^{a}(a+b)^{b-\frac{1}{2}}}{b^{a}\ b^{b-\frac{1}{2}}}$$

$$= \frac{e^{-a}}{b^{a}}\ (a+b)^{a}\ \left(\frac{a+b}{b}\right)^{b-\frac{1}{2}}$$

Next, consider the logarithmic series

$$-\log(1-x) = x+\frac{x^{2}}{2}+\frac{x^{3}}{3}+\ldots$$

$$\log\left(1+\tfrac{a}{b}\right)^{b} = b\log\left(1+\tfrac{a}{b}\right)$$

$$= b\log\left(1-\left(-\tfrac{a}{b}\right)\right)$$

$$-\log\left(1+\tfrac{a}{b}\right)^{b} = -b\log\left(1-\left(-\tfrac{a}{b}\right)\right)$$

$$= b\log\left(1-\left(-\tfrac{a}{b}\right)\right)$$

$$= \left\{\left(-\frac{a}{b}\right)+\frac{\left(-\frac{a}{b}\right)^{2}}{2}+\frac{\left(-\frac{a}{b}\right)^{3}}{3}+\ldots\right\}$$

$$=b\left\{-\frac{a}{b}+\frac{a^2}{2b^2}-\frac{a^3}{3b^3}+\dots\right\}$$

$$=b\left\{-a+\frac{a^2}{2b}-\frac{a^3}{3b^2}+\frac{a^4}{4b^3}-\dots\right\}$$

$$\lim_{b\to\infty}\left(1+\frac{a}{b}\right)^b=e^a$$

Also

$$\log\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]^{-(a+b)}=\log\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]^{-a}+\log\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]^{-b}$$

$$=-\log\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]-b\log\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]$$

$$=a\left\{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)+\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^2+\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^3+\dots\right\}+b\left\{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)+\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^2+\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^3+\dots\right\}$$

Therefore

$$\left[1+\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right]^{-(a+b)}$$

$$=\exp\left(a\left\{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)+\frac{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^2}{2}+\frac{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^3}{3}+..\right\}+b\left\{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)+\frac{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^2}{2}+\frac{\left(-\tfrac{1}{b}\left(\tfrac{\theta}{\beta}\right)^c\right)^3}{3}+..\right\}\right)$$

Therefore

$$\lim_{b\to\infty}\left(1+\frac{\left(\frac{\theta}{\beta}\right)^c}{b}\right)^{-a}\lim_{b\to\infty}\left(1+\frac{\left(\frac{\theta}{\beta}\right)^{-b}}{b}\right)=e^{-a}\times e^{-\left(\frac{\theta}{\beta}\right)^c}$$

Therefore

$$g(\theta)=\frac{|C|\,\theta^{ca-1}}{\beta\Gamma a}\left(\frac{a+b}{b}\right)^a\left(\frac{a+b}{b}\right)^{b-\frac{1}{2}}e^{-a}\times e^a\times e^{-\left(\frac{\theta}{\beta}\right)^c}$$

$$\lim_{b\to\infty}g(\theta)=\frac{|C|\,\theta^{ca-1}}{\beta\Gamma a}\lim_{b\to\infty}\left(\frac{a+b}{b}\right)^a\lim_{b\to\infty}\left(\frac{a+b}{b}\right)^{b-\frac{1}{2}}e^{-a}\times e^a\times e^{-\left(\frac{\theta}{\beta}\right)^c}$$

$$g(\theta)=\frac{|C|\,\theta^{ca-1}}{\beta\Gamma a}e^{-\left(\frac{\theta}{\beta}\right)^c}\qquad \theta>0\qquad \beta,|C|,a>0\qquad (2)$$

which is called generalized-Gamma distribution.

2.2 Binomial-Generalized Gamma distribution

The binomial-generalized gamma distribution is given by

$$f(x)=\binom{n}{x}\int_0^\infty\frac{\theta^x}{(1+\theta)^n}\frac{|C|\left(\frac{\theta}{\beta}\right)^{ca-1}}{\beta\Gamma a}e^{-\left(\frac{\theta}{\beta}\right)^c}d\theta$$

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$$= \frac{\binom{n}{x}|C|}{\beta^{ca}\Gamma a}\int_0^\infty \frac{\theta^{x+ca-1}}{(1+\theta)^n} e^{-\left(\frac{\theta}{\beta}\right)^c} d\theta$$

$$= \frac{\binom{n}{x}|C|}{\beta^{ca}\Gamma a}\sum_{k=0}^{\infty}\binom{-n}{x}\int_0^\infty \theta^{k+x+ca-1} e^{-\left(\frac{\theta}{\beta}\right)^c} d\theta$$

Let $y = \left(\frac{\theta}{\beta}\right)^c$, $\quad \frac{dy}{d\theta} = \frac{C\theta^{c-1}}{\beta^c}$, $\quad d\theta = \frac{\beta^c dy}{C\theta^{c-1}}$, $\quad \theta = \beta y^{\frac{1}{c}}$

$$= \frac{\binom{n}{x}|C|}{\beta^{ca}\Gamma a}\sum_{k=0}^{\infty}\binom{-n}{k}\int_0^\infty \left(\beta y^{\frac{1}{c}}\right)^{k+x+ca-1} e^{-y}\,\frac{\beta^c dy}{C\left(\beta y^{\frac{1}{c}}\right)^{c-1}}$$

$$= \frac{\beta^{x+k}\binom{n}{x}}{\Gamma a}\sum_{k=0}^{\infty}\binom{-n}{k}\int_0^\infty y^{\frac{k+x}{c}+a-1} e^{-y} dy$$

Therefore

$$f(x) = \binom{n}{x}\sum_{k=0}^{\infty}\binom{-n}{k}\frac{\beta^{x+k}}{\Gamma a}\Gamma\left(a + \tfrac{x+k}{c}\right) \qquad (3)$$

2.3 Recursive form of Binomial-Generalized Gamma distribution

The binomial-generalized Gamma recursive model can be obtained using the ratio method. Consider a ratio of two consecutive probabilities.

$$f(x) = \binom{n}{x}\sum_{k=0}^{\infty}\binom{-n}{k}\frac{\beta^{x+k}}{\Gamma a}\Gamma\left(a + \tfrac{x+k}{c}\right) \qquad \text{and}$$

$$f(x-1) = \binom{n}{x-1}\sum_{k=0}^{\infty}\binom{-n}{k}\frac{\beta^{x+k-1}}{\Gamma a}\Gamma\left(a + \tfrac{x+k-1}{c}\right)$$

Then

$$\frac{f(x)}{f(x-1)} = \frac{\binom{n}{x}\sum_{k=0}^{\infty}\binom{-n}{k}\frac{\beta^{x+k}}{\Gamma a}\Gamma\left(\tfrac{x+k}{c}+a\right)}{\binom{n}{x-1}\sum_{k=0}^{\infty}\binom{-n}{k}\frac{\beta^{x+k-1}}{\Gamma a}\Gamma\left(\tfrac{x+k-1}{c}+a\right)}$$

$$= \frac{n!(n-x+1)!(x-1)!\beta^{x+k}\Gamma\left(\tfrac{x+k}{c}+a\right)}{(n-x)!x!n!\beta^{x+k-1}\Gamma\left(\tfrac{x+k-1}{c}+a\right)}$$

$$= \frac{(n-x+1)\beta\left(\tfrac{x+k}{c}+a-1\right)!}{x\left(\tfrac{x+k-1}{c}+a-1\right)!}$$

$$\frac{f(x)}{f(x-1)} = \frac{(n-x+1)\beta(x+k+ac-c)}{x}$$

Therefore

$$[x]f(x)=\beta[(n-x+1)(x+k+ac-c)]f(x-1) \qquad (4)$$

2.4 Panjer-Willmot recursive model

Panjer and Willmot (1982) came up with the following recursive model;

$$f(x)\sum_{t=0}^{k}a_t x^t = f(x-1)\sum_{t=0}^{k}b_t(x+1)^t$$

$$f(x)\left[a_0+\sum_{t=1}^{k}a_t x^t\right]= f(x-1)\left[b_0+\sum_{t=1}^{k}b_t\left(x-1\right)^t\right]$$

Therefore

$$a_0 f(x)+\sum_{t=1}^{k}a_t x^t f(x)=b_0 f(x-1)+\sum_{t=1}^{k}b_t\left(x-1\right)^t f(x-1) \qquad x=0,1,2...$$

Where

$$x^{(t)}=x(x-1)(x-2)....(x-t+1)$$

$$(x-1)^{(t)}=(x-1)(x-2)....(x-t)$$

$$x^{(0)}=(x-1)^0=1$$

When $k=2$ the recursive model is

$$[a_0+a_1x+a_2x(x-1)]f(x)=[b_0+b_1(x-1)+b_2(x-1)(x-2)]f(x-1) \qquad x=1,2.... \qquad (5)$$

To obtain differential equation (4) is multiplied by s^x and then the result is summed over x.

$$a_0\sum_{x=1}^{\infty}f(x)s^x+\sum_{x=1}^{\infty}\sum_{t=1}^{k}a_t x^t f(x)s^x=b_0\sum_{x=1}^{\infty}f(x-1)s^x+\sum_{x=1}^{\infty}\sum_{t=1}^{k}b_t(x-1)^t f(x-1)s^x$$

$$a_0\sum_{x=1}^{\infty}f(x)s^x+\sum_{t=1}^{k}\left[\sum_{x=1}^{\infty}a_t x^t f(x)s^x\right]=b_0\sum_{x=1}^{\infty}f(x-1)s^x+\sum_{t=1}^{k}\left[\sum_{x=1}^{\infty}b_t(x-1)^t f(x-1)s^x\right] \qquad (6)$$

Define

$$G(s)=\sum_{x=0}^{\infty}f(x)s^x$$

$$G^{'}(s)=\sum_{x=1}^{\infty}xf(x)s^{x-1}$$

$$G^{''}(s)=\sum_{x=2}^{\infty}x(x-1)f(x)s^{x-2}$$

$$G^{'''}(s)=\sum_{x=3}^{\infty}x(x-1)(x-2)f(x)s^{x-3}$$

$$\vdots$$

$$G^{(t)}(s)=\sum_{x=t}^{\infty}x(x-1)(x-2)...(x-t+1)f(x)s^{x-t}$$

i.e

$$G^{(t)}(s)=\sum_{x=t}^{\infty}x^{(t)}f(x)s^{x-t} \qquad (7)$$

Therefore equation (5) becomes

$$a_0\left[G(s)-f(0)\right]+\sum_{t=1}^{k}a_t s^t G^{(t)}(s)=b_0 s\sum_{x=1}^{\infty}f(x-1)s^{x-1}+\sum_{t=1}^{k}\left[b_t\sum_{x=1}^{\infty}(x-1)^t f(x-1)\ s^x\right]$$

Also defining

$$G(s)=\sum_{x=1}^{\infty}f(x-1)s^{x-1}$$

$$G^{'}(s)=\sum_{x=2}^{\infty}(x-1)f(x-1)s^{x-2}$$

$$G^{''}(s)=\sum_{x=3}^{\infty}(x-1)(x-2)f(x-1)s^{x-3}$$

$$G^{'''}(s)=\sum_{x=4}^{\infty}(x-1)(x-2)(x-3)f(x-1)s^{x-4}$$

$$G^{(t)}(s)=\sum_{x=t+1}^{\infty}(x-1)^t\ f(x-1)s^{x-t-1} \qquad (8)$$

Therefore equation (5) becomes

$$a_0G(s)-a_0f(0)+\sum_{t=1}^{k}a_t s^t G^{(t)}(s)=b_0sG(s)+\sum_{t=1}^{k}b_t s^{t+1}G^{(t)}(s)$$

The corresponding differential equation in probability generating function is

$$a_0G(s)-a_0f(0)+a_1sG^{'}(s)+a_2s^2G^{''}(s)=b_0sG(s)+b_1s^2G^{'}(s)+b_2s^3G^{''}(s)$$

$$s^2(a_2-b_2s)G^{''}(s)+s(a_1-b_1s)G^{'}(s)+(a_0-b_0s)G(s)=a_0f(0) \qquad (9)$$

2.5 Hesselager's Recursive model

Hesselager's (1992) class of recursive models is given by

$$f(x)\sum_{t=0}^{k}b_t x^t = f(x-1)\sum_{t=0}^{k}a_t x^t \qquad x=0,1,2...$$

for some positive integer k.

$$\therefore\ f(x)\sum_{t=0}^{k}b_t x^t = f(x-1)\sum_{t=0}^{k}a_t(x-1+1)^{(t)}$$

$$= f(x-1)\sum_{t=0}^{k}a_t\left[\sum_{l=0}^{t}\binom{t}{l}(x-1)^l\right]$$

$$= f(x-1)\sum_{t=0}^{k}\sum_{l=0}^{k}a_t\binom{t}{l}(x-1)^l$$

$$\therefore\ f(x)\sum_{t=0}^{k} b_t x^t = f(x-1)\sum_{l=0}^{k} C_l (x-1)^l$$

$$= f(x-1)\sum_{l=0}^{k} C_l (x-1)^l$$

$$\text{where} \quad C_l = \sum_{t=0}^{k} a_t \binom{t}{l}$$

Putting $k = 2$, we have

$$\left[b_0 + b_1 x + b_2 x^2\right] f(x) = \left[C_0 + C_1 (x-1) + C_2 (x-1)^2\right] f(x-1) \qquad (10)$$

Where

$$C_l = \sum_{t=0}^{2} a_t \binom{t}{l}$$

$$C_0 = \sum_{t=0}^{2} a_t \binom{t}{0} = a_0 + a_1 + a_2$$

$$C_1 = \sum_{t=0}^{2} a_t \binom{t}{1} = a_1 + 2a_2$$

$$C_2 = \sum_{t=0}^{2} a_t \binom{t}{2} = a_2$$

To determine corresponding differential equation we multiply (8) by s^x and sum the result over x.

$$\sum_{x=1}^{\infty}\left(b_0 + b_1 x + b_2 x^2\right) f(x) s^x = \sum_{x=1}^{\infty}\left(C_0 + C_1 (x-1) + C_2 (x-1)\right)^2 f(x-1) s^x$$

$$b_0 \sum_{x=1}^{\infty} f(x) s^x + b_1 \sum_{x=1}^{\infty} x f(x) s^x + b_2 \sum_{x=1}^{\infty} x^2 f(x) s^x$$

$$= C_0 \sum_{x=1}^{\infty} f(x-1) s^x + C_1 \sum_{x=1}^{\infty} (x-1) f(x-1) s^x + C_2 \sum_{x=1}^{\infty} (x-1)^2 f(x-1) s^x$$

$$b_0\left(G(s) - f(0)\right) + b_1 s G'(s) + b_2 \sum_{x=1}^{\infty} x^2 f(x) s^x$$

$$= C_0 s G(s) + C_1 s^2 G''(s) + C_2 \sum_{x=1}^{\infty} (x-1)^2 f(x-1) s^x$$

$$b_0\left[G(s) - f(0)\right] + b_1 s G'(s) + b_2 \sum_{x=1}^{\infty} x(x-1+1) f(x) s^x$$

$$= C_0 s G(s) + C_1 s^2 G'(s) + C_2 \sum_{x=1}^{\infty} (x-1)(x-2+1) f(x-1) s^x$$

$$b_0\left[G(s) - f(0)\right] + b_1 s G'(s) + b_2 \sum_{x=1}^{\infty} x(x-1) f(x) s^x + b_2 \sum_{x=1}^{\infty} x f(x) s^x$$

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$$= C_0 sG(s) + C_1 s^2 G^{'}(s) + C_2 \sum_{x=1}^{\infty} (x-1)(x-2) f(x-1) s^x + C_2 \sum_{x=1}^{\infty} (x-1) f(x-1) s^x$$

$$b_0 \left[G(s) - f(0) \right] + b_1 sG^{'}(s) + b_2 s^2 G^{''}(s) + b_2 sG^{'}(s) = C_0 sG(s) + C_1 s^2 G^{'}(s) + C_2 s^3 G^{''}(s) + C_2 s^2 G^{'}(s)$$

$$s^2 (b_2 - C_2 s) G^{''}(s) + \left[(b_1 + b_2)s - (C_1 + C_2)s^2 \right] G^{'}(s) + (b_0 - C_0 s) G(s) = b_0 f(0) \qquad (11)$$

3.0 Results

3.1 Panjer-Willmot recursive model

From binomial-generalized Gamma recursive model which is

$$xf(x) = \beta \left[(n - x + 1)(x + k + ac - c) \right] f(x-1)$$

The equation can be re-written as

$$xf(x) = \beta \left[(n - (x-1))((k + ac - c + 1) + (x-1) \right] f(x-1)$$

$$xf(x) = \beta \left[n(k + ac - c + 1) - (x-1)(k + ac - c + 1) + n(x-1) - (x-1)^2 \right] f(x-1)$$

$$xf(x) = \beta \left[n(k + ac - c + 1)(n - k - ac + c - 1)(x-1) - (x-1)(x - 2 + 1) \right] f(x-1)$$

$$xf(x) = \beta \left[n(k + ac - c + 1)(n - k - ac + c - 2)(x-1) - (x-1)(x-2) \right] f(x-1)$$

which is Panjer-Willmot recursive model with parameters;

$$k = 2, \qquad a_0 = 0 \qquad a_1 = 1, \qquad a_2 = 0$$

$$b_0 = \beta n(k + ac - c + 1), \qquad b_1 = \beta(n - k - ac + c - 2), \qquad b_2 = -\beta$$

The corresponding differential equation in probability generating function is

$$s^2 \left[a_2 - b_2 s \right] G^{''}(s) + s(a_1 - b_1 s) G^{'}(s) + (a_0 - b_0 s) G(s) = a_0 f(0)$$

$$s^2 \left[0 + \beta s \right] G^{''}(s) + s(1 - \beta(n - k - ac + c - 2)s) G^{'}(s) + (0 - \beta n(k + ac - c + 1) sG(s) = 0$$

$$s^2 \beta G^{''}(s) + \left[1 - \beta(n - k - ac + c - 2)s \right] G^{'}(s) - \beta n(k + ac - c + 1) G(s) = 0$$

3.2 Hesselager's Recursive model

Binomial-Generalized Gamma recursive model is

$$xf(x) = \beta \left[(n - x + 1)(x + k + ac - c) \right] f(x-1)$$

$$xf(x) = \beta \left[(n - (x-1))(k + ac - c + 1) + (x-1) \right] f(x-1)$$

$$xf(x) = \beta \left[n(k + ac - c + 1) + n(x-1) - (k + ac - c + 1)(x-1) - (x-1)^2 \right] f(x-1)$$

$$xf(x) = \beta \left[n(k + ac - c + 1) + (n - k - ac + c - 1)(x-1) - (x-1)^2 \right] f(x-1)$$

which is Hesselager's recursive model with parameters;

$$k = 2, \qquad b_0 = 0, \qquad b_1 = 1 \qquad b_2 = 0$$

$$C_0 = \beta n(k + ac - c + 1), \qquad C_1 = \beta(n - k - ac + c - 1), \qquad C_2 = -\beta$$

The corresponding differential equation is

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$$s^2(b_2 - C_2 s)G''(s) + [(b_1 + b_2) - (c_1 + c_2)s]G'(s) + (b_0 - C_0 s)G(s) = b_0 f(0)$$
$$s^2(0 - \beta s)G''(s) + s[1 - \beta(n - k - ac + c - 2)s]G'(s) + (0 - \beta n(k + ac - c + 1)sG(s) = 0$$
$$s^2 \beta G''(s) + s[1 - \beta(n - k - ac + c - 2)]G'(s) - \beta n(k + ac - c + 1)G(s) = 0$$

4.0 Discussion

From the discussion above it is noted that generated Gamma mixing distribution is a special case of generalized beta distribution of the second kind with four parameters. The mixture is obtained by applying Stirling formula to Gamma function for large values of parameter b and also by through application of logarithmic series for large values of parameter b. The mixture forms a recursive relation which can be fitted to both Panjer-Willmot and Hesselager's recursive models. It is noted that different parameters are obtained from the mixture when fitted to different recursive models. However the differential equations in probability generating functions obtained are similar indicating that different recursive models can be fitted effectively to the same distribution.

5.0 Conclusion and recommendations

Probability distribution can also be constructed by using recursive relations in probabilities. The beta-binomial mixture derived its recursive relation by applying ratio method. The recursive model so obtained fitted well into the recursive models of Panjer-Willmot (1982) and Hesselager's(1992) class of recursive models. And their corresponding differential equations obtained. Thus the mixed distributions of binomial and beta distribution with its generalizations can be expressed in terms of recursive relations and can as well generate differential equations. However the equations so obtained with k > 1 seems to be difficult to determine their respective moments,mean and variances. This can be looked at on further research.

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